

METHOD FOR PREDICTING RIVER ICE FORMATION

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ABSTRACT

Ice formation on a water body occurs in response to a heat exchange between the water surface and the ambient air resulting in an upward net heat loss from the water surface. For a river situation, all short periods of warming and cooling prior to ice formation affect this heat exchange and consequently the date of river ice formation. To incorporate the response of a river to these periods, numerical constants are derived from a relationship between daily mean air temperatures and previously observed dates of ice formation using the theory of a weighted mean temperature.

Two sets of curves are developed which can be used to forecast the dates of: 1) first permanent ice; and 2) complete freeze-over from shore to shore on the St. Marys River at Sault Ste. Marie, Michigan. By applying these curves, a day-to-day forecast of the date of ice formation is provided based on expected departure-from-normal air temperatures.

1. Introduction

River ice has been widely recognized as an important deterrent to the extension of winter navigation in the Great Lakes-St. Lawrence Seaway system. Considerable research has been conducted in ice forecasting, most of which has been in the exposed part of the lake system, where the bulk of shipping operates. Lake navigation, however, is not only restricted by drifting pack ice on the open lakes, and thick harbor ice, but also in the constricted sections of the interlake system. The St. Marys River at Sault Ste. Marie, Michigan is one of these critical areas because it is closed to navigation, at times, in late fall to allow the ice to form behind the booms of the Soo Locks.

There are three major situations in which ice becomes a problem to lock operations: 1) when entranceways to the locks fill with ice which must be discharged before vessels can proceed; 2) when the ice thickness on lock walls and gates increases as the water level is raised and lowered with the passage of each vessel; and 3) when gate operations are obstructed by floating ice. To minimize some of these ice effects before winter close-down, air-bubbling systems are used in the locks, heaters on the lock gates, and polyethylene foam is spread over the ice surface to accelerate melting. Although these efforts to extend the navigation season have proved satisfactory, more precise information on the time of ice formation would be economically beneficial. The closing dates currently operative are primarily due to administrative decisions made weeks in advance, and represent a subjective rather than a scientifically reliable method to forecast the time of ice formation.

An objective approach to predict ice formation for the Soo Locks on the St. Marys River at Sault Ste. Marie, Michigan, would provide better basic information necessary for skillful planning and decision making. The scientific approach described in this investigation is based on the hypothesis that the dates of first permanent ice and complete freeze-over can be computed by using the theory of a weighted-mean air temperature.

The term "first permanent ice", defines the date on which ice first forms on the water surface and does not thereafter melt completely, until spring break-up of the following year. A small amount of ice first forming on a given date, but melting again completely a few days later does not constitute the date of first permanent ice. The term "complete freeze-over", defines the date on which the water body was reported to be completely covered except near outlets of power or industrial plants which usually remain ice-free all winter.

Numerical constants are derived from a relationship between

mean air temperatures and previously observed dates of ice formation, and these constants are used to develop two sets of curves from which the dates of first permanent ice and complete freeze-over can be forecast.

2. Review of the Literature

A review of the literature reveals that various ice forecasting methods have been developed. These methods range from the use of freezing degree days and heat budgets of the water body, to very complicated empirical equations that incorporate numerous meteorological and hydrological parameters.

Richards (1964) used degree days to predict ice on the Great Lakes. In his study, cumulative thawing degree days were used as an index of antecedent heating prior to ice formation and cumulative freezing air temperatures were used as an index of winter severity. Ice cover data were obtained by aerial reconnaissance and air temperatures, recorded at certain locations on the Great Lakes, were used to predict ice formation. Noble (1965) employed water temperature data to calculate the heat stored in Lake Michigan during the winter season 1962/63 (when the lake was totally covered by ice) and 1963/64 (when the lake was nearly ice-free). Although his data span an interval too short to allow an extensive investigation, Noble's hypothesis is that the prediction of ice could be attempted by using fall lake temperatures from the deep layers in conjunction with cumulative degree days according to the method described by Richards (1964). Shuliakovskii (1966) also suggests the use of air temperature forecasts for the calculation of the appearance of ice on rivers; however, the results of his mathematical approach in actual application were not developed. Poulin, Robinson and Witherspoon (1971) employed a method to calculate the cooling of water from Lake Ontario as it flowed down the St. Lawrence River between Kingston, Ontario and Sorel, Quebec. An empirical formula, developed by the Joint Board of Engineers on the St. Lawrence Waterways Project, was used to produce synthetic water-surface temperatures with respect to time and discharge. Using this method, water-surface temperature computations beginning 1 November for the winter season 1959/60 through 1968/69 were continued to freeze-over at Beauharnois, Cornwall and Montreal. Here, freeze-over was defined as the date on which the water-surface temperature fell below 32.5°F. Computed dates of freeze-over differed from the mean observed dates of freeze-over at these locations by standard deviations of 2.4 days, 5.3 days and 7.1 days, respectively.

From the above discussion, it is apparent that the ice forecasting methods suggested by Richards (1964) and Noble (1965) could produce a reasonably consistent average relationship based on after-the-fact data, but variations on either side of

such a relationship would be large. Poulin, Robinson and Witherspoon (1971) proved that it is possible to reduce these variations from the average. However, their method, and the methods suggested by Shuliakovskii (1966) require a substantial amount of data and would prove awkward in application. A fundamental omission in the methods reviewed is that they do not consider the rapid response of a water body to short periods of warming or cooling just prior to ice formation, which would obviously skew the final forecast. One approach to the problem presented by these short periods is based on the theory of a weighted mean air temperature. Such an approach incorporates the response of a river to rapidly changing weather conditions.

Since Rhode (1952) developed his theory of a weighted mean air temperature, considerable progress has been made in determining the date of ice formation by utilizing air temperatures. This theory is logically based upon a simplified physical law of heat flow between a water surface and the air. Rhode used the formula of iterations;

$$\tau_n = \tau_{n-1} + (1 - e^{-k\Delta t}) (T_n - \tau_{n-1})^1$$

where T is the air temperature, τ the water-surface temperature, t time and k^{-1} is a constant with inverse dimensions of time. In order to demonstrate that ice prediction could be done with a single variable, air temperature, Rhode based his investigation on observations in the winter seasons 1937/38 through 1949/50 at Gálve Road, the southernmost harbor in the Gulf of Bothnia. Utilizing the weighted-mean air temperature formula, Rhode found that ice in this area formed within ± 1 day of the computed date in 56 percent of the years and within ± 2 days in 87 percent of the years.

Bilello (1961) employed the theory of a weighted-mean air temperature and the associated empirical formula developed by Rhode (1952) for the prediction of sea-ice formation in the Canadian Arctic Archipelago. He found that except in two years, this method yielded freeze-over dates within ± 3 days of the observed dates. In a later report, Bilello (1964a) developed ice prediction curves for both first permanent ice and complete freeze-over for one bay, ten lakes and seventeen river locations throughout Canada, by employing the same method. In an abridged version of this report, Bilello (1964b) reported on the Mackenzie River at Fort Good Hope, Canada. From observations of the winter seasons 1941/42 through 1957/58, two sets of curves were developed which were used to forecast the dates of first permanent ice and complete freeze-over from shore to shore. Computed dates of both first permanent ice and complete freeze-over were within ± 3 days of the observed dates.

Using the computational procedure outlined by Bilello, Williams (1965) produced synthetic water-surface temperatures from daily mean air temperatures for McKay Lake, a small, sheltered lake in Ottawa, Canada. From observations of the winter seasons 1959/60 through 1963/64, Williams predicted dates of freeze-over that were correct within ± 3 days, of the observed dates, similar to that obtained by Bilello.

The results of these various investigations indicate that the theory of a weighted-mean air temperature developed by Rhode (1952) is the best method to obtain reliable estimates on the dates of ice formation.

3. Development of the Ice Forecast Procedure

In this study, the theory of a weighted mean temperature and the associated empirical formula as developed by Rhode (1952) is applied to the St. Marys River at Sault Ste. Marie, Michigan. The data span eleven years and include the winter season 1962/63 through 1973/74 except for 1965/66, when ice data were not observed.

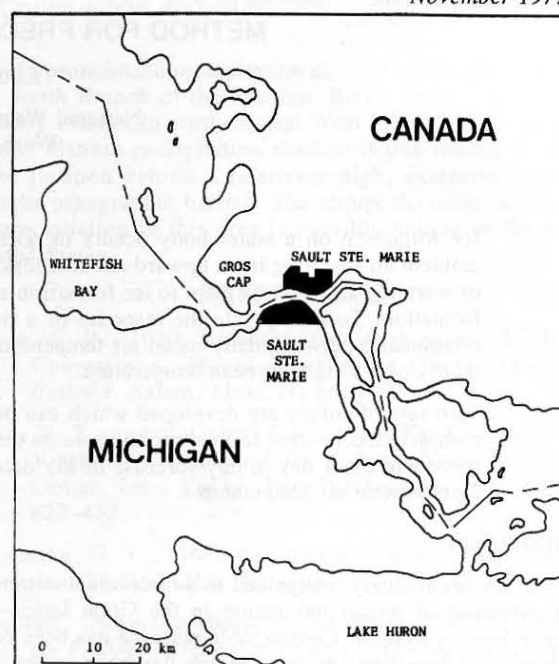


FIG. 1. The Saint Marys River.

The St. Marys River (Figure 1) originates in Whitefish Bay, the easternmost part of Lake Superior, and flows southeastward between Michigan and Ontario and into northern Lake Huron. At the site under investigation, the river is approximately 3-kilometers wide, has a mean depth of approximately 7 meters, and an average annual discharge of 2,250 cubic meters per second. Observations of daily mean air temperatures during the eleven years studied were recorded by the National Weather Service office at Sault Ste. Marie Municipal Airport, which is approximately 2.5-kilometers south and inland from the St. Marys River. The dates of first permanent ice and complete freeze-over in the stretch between the Soo Locks and the headwaters at Gros Cap in these years were published by the Canadian Meteorological Service (1971).

Rhode's formula provides a method by which the date of ice formation can be computed by using past air temperatures. The procedure is to determine the value of k^{-1} in the expression $1 - e^{-k\Delta t}$ by trial-and-error. The condition which must be met is a coordination of the date of ice formation with the k^{-1} solution of the equation received when $\delta\tau = 0$ and $\tau=0$. By associating the time when the air temperature sequence reduces to the freezing point with the date of ice formation, the best fitting " β " function is determined. If, for example, $k^{-1} = 0.051$ is the solution to the equation, it can be shown that $e^{-k\Delta t} = -0.051\Delta t = 0.950$ were $\Delta t = 1$. It follows that $1 - e^{-k\Delta t} = 0.050$, and this value (0.050) is termed the β function. The primary purpose of the β function is to permit a means of reproducing a daily account of the heat budget of the water body as derived from air temperatures.

According to the theory of a weighted mean temperature, if the initial time is chosen as a point far enough in the past, the water-surface temperature essentially becomes equal to the weighted mean of all past air temperatures. The initial value used in this study is the mean air temperature for the month of June. In the computational procedure, this initial value is subtracted from the daily mean air temperature for 1 July. This value is then multiplied by the β function and this weighted correction is added to the initial value. This weighted mean temperature is subtracted from the daily mean air temperature for 2 July. This value is again multiplied by the β function and

¹ The full mathematical derivation of the theory is presented in the

TABLE 1

Daily Tabulation of Weighted Mean Temperature Computations

A	B	C	D	E	F
Date	Daily Mean Air Temp (°C)	(F - 1 Day)	(B - C) Mean For Month of June	(D × 0.050)	Weighted Mean Temp (°C) (C + E)
July 1	17.2	14.92	2.28	.11	14.92
July 2	17.5	15.04	2.46	.12	15.04
July 3	18.9	15.16	3.74	.19	15.35
July 4	18.3	15.35	2.95	.15	15.50
July 5	20.6	15.50	5.10	.26	15.76
July 6	20.3	15.76	4.54	.23	15.99
July 7	23.6	15.99	7.61	.38	16.37
July 8	17.2	16.37	.38	.04	16.41
July 9	13.6	16.41	-2.81	-.14	16.27
July 10	15.0	16.27	-1.27	-.06	16.21
July 11	17.8	16.21	1.59	.08	16.29
July 12	15.3	16.29	-.99	-.05	16.24
July 13	16.9	16.24	.66	.03	16.27
July 14	13.1	16.27	-3.17	-.16	16.11
July 15	15.6	16.11	-.51	-.03	16.09

this weighted correction is added to the previously computed weighted mean temperature, and the procedure is repeated. The job is repetitious and consequently ideal for computer application. Given $k^{-1} = 0.051$ and $\Delta t = 1$, the equation provides a daily tabulation of weighted mean temperature computations as shown in Table 1. It should be noted that B, C, D, E and F are the equivalents of T_n , τ_{n-1} , $T_n - \tau_{n-1}$, $(1 - e^{-k\Delta t}) (T_n - \tau_{n-1})$ and τ_n , respectively.

These computations are continued to the dates of first permanent ice and complete freeze-over, thus incorporating the response of a river to all short periods of warming or cooling prior to ice formation. If on the date of first permanent ice, or complete freeze-over, the weighted mean temperature in Column F is less than or equal to 0°C, the β function for that winter season is the one which is required according to the thermal characteristics existing in the data. If, however, the weighted mean temperature in Column F is a positive value on the observed date of either first permanent ice or complete freeze-over, then the computer program, as designed, will select a lower value for the β function to alter the rate of change until the weighted mean temperature in Column F becomes less than or equal to 0°C. In this study, a β function was computed for seven randomly selected winter seasons at Sault Ste. Marie, Michigan. Once a β function has been established for each winter season, the β functions are averaged to determine a final

β function for first permanent ice and complete freeze-over. Table 2 shows the winter seasons used to compute β , the observed dates of first permanent ice and complete freeze-over and the average β function obtained in each case.

Quite different thermal implications are provided by the β functions in each winter season. If, for instance, a short period of warming occurs just prior to ice formation, the value of the β function will be smaller than if a short period of cooling had occurred. Figure 2 shows three air temperature curves beginning 10 November to first permanent ice for the winter season 1962/63: 1) normal (based on the interval 1941-1970), 2) daily mean and 3) weighted mean. The course of weighted mean temperatures parallels that of the daily mean temperatures, except for all short periods of warming and cooling prior to first permanent ice. It is interesting to note that the course of below freezing temperatures occurring after 5 December resulted in first permanent ice on 11 December rather than the mean date, 15 December, based on the eleven winter seasons used in this study. The 30-year normal temperature curve appears far below 0°C prior to first permanent ice since it does not incorporate any influence of earlier warmer temperatures.

By applying the average β functions and utilizing daily normals of temperature (1941-1970) at Sault Ste. Marie, Michigan, ice forecast curves can be developed for first permanent ice ($\beta =$

TABLE 2
 β Functions For Sault Ste. Marie

Winter Season	First Permanent Ice Observed Date	β	Complete Freeze-Over Observed Date	β
1962/63	11 December	0.050	13 January	0.016
1963/64	11 December	0.063	13 January	0.017
1967/68	24 December	0.027	23 January	0.012
1968/69	29 December	0.020	06 February	0.011
1970/71	06 December	0.057	31 January	0.012
1971/72	04 January	0.020	23 January	0.014
1973/74	06 December	0.020	30 January	0.012
	Average	0.037	Average	0.013

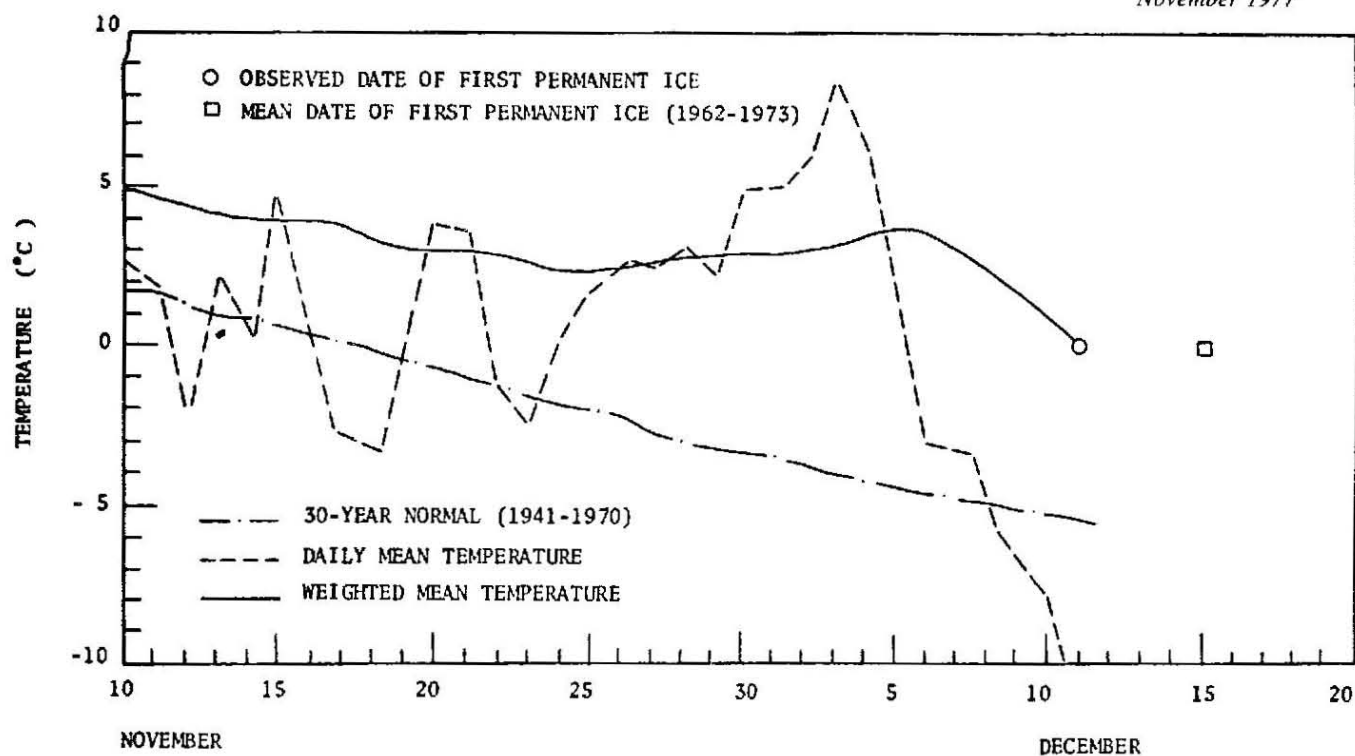


FIG. 2. Comparison of normal, daily mean and weighted mean temperature curves at Sault Ste. Marie during the winter season 1962/63.

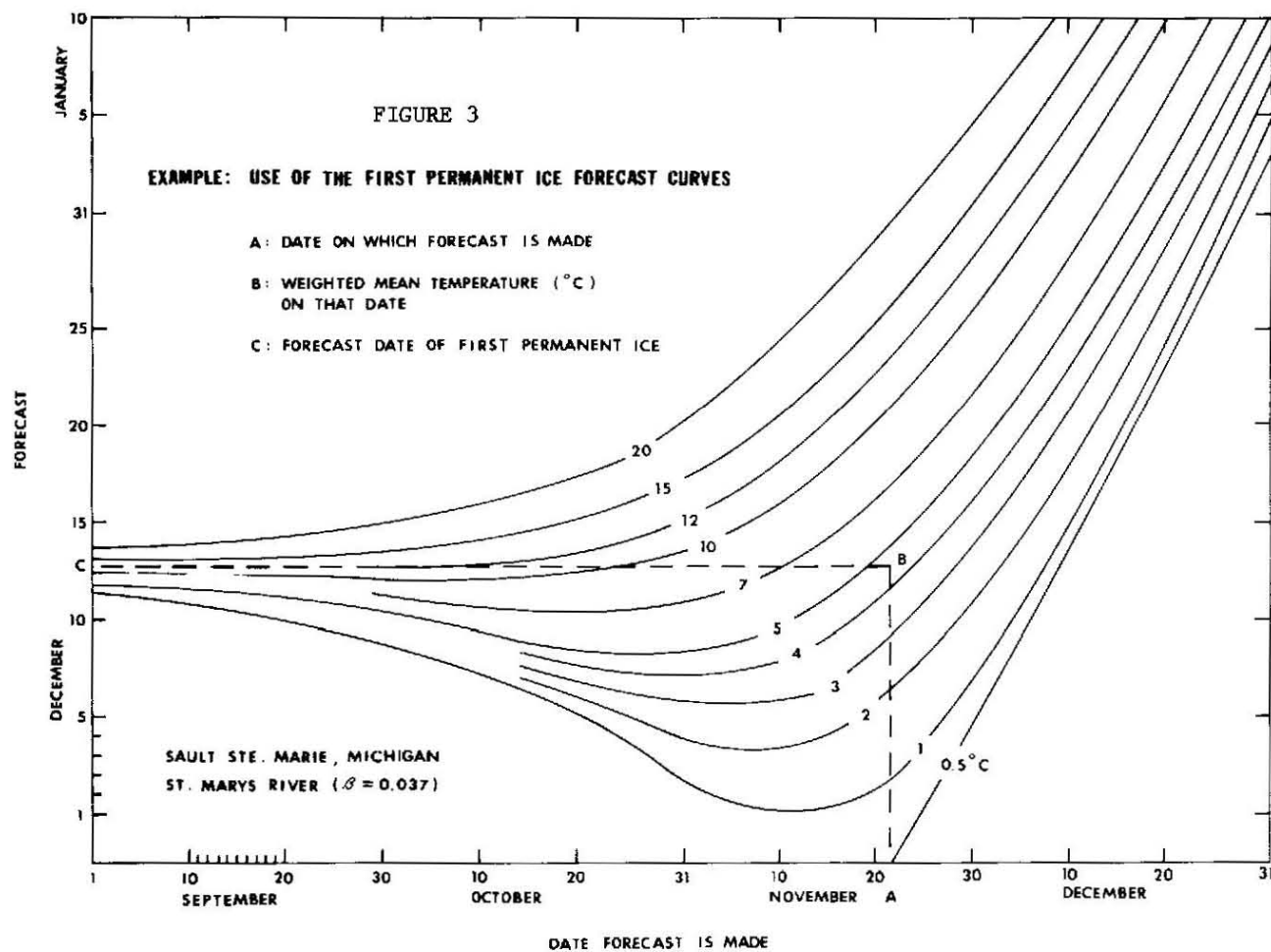


FIG. 3. Use of the first permanent ice forecast curves.

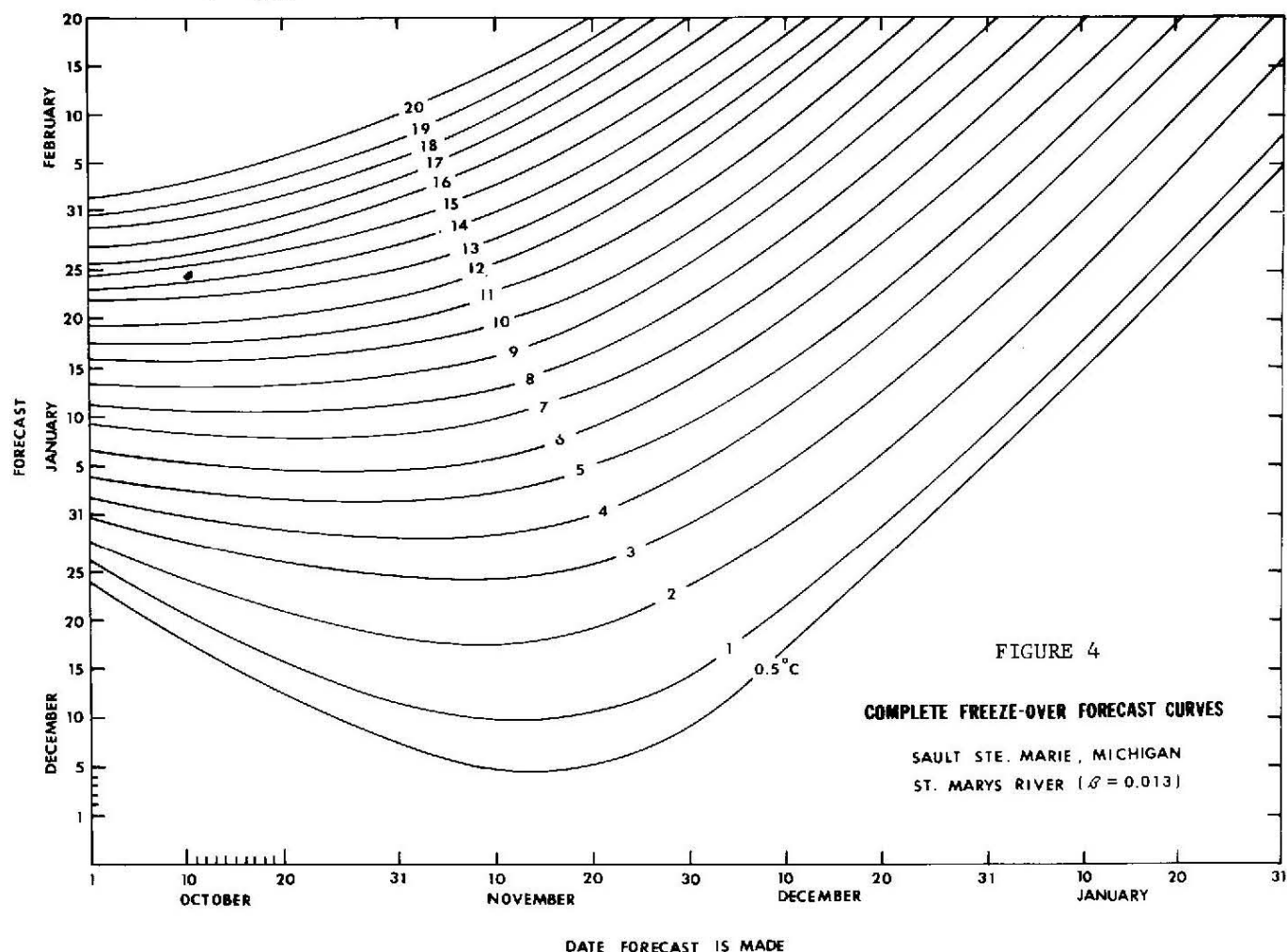


FIG.4. Complete freeze-over forecast curves.

0.037) and complete freeze-over ($\beta = 0.013$). A series of curves were constructed for first permanent ice and complete freeze-over using the date of the weighted mean temperature along the x-axis and the number of days after 30 June that the weighted mean temperature reduced to 0°C or less along the y-axis. Smooth curves were drawn between the points of equal weighted mean temperature values to provide the required forecast curves. A series of forecast curves are essential since weighted mean temperature values may vary from 0°C to 15°C and above, depending on how far in advance an ice forecast is desired. The curves for both first permanent ice and complete freeze-over at Sault Ste. Marie, Michigan, appear in Figure 3 and Figure 4, respectively.

As an example in the use of this method for the forecast of first permanent ice: if the weighted mean temperature at Sault Ste. Marie on 22 November is 4.72°C then first permanent ice, based on normal temperatures, is forecast for 12 December as shown in Figure 3. However, if on 22 November the daily air temperatures for the next two weeks are expected to average 5.10°C below normal then, using the computation described in Table 1, the forecast of first permanent ice becomes 5 December as shown in Table 3.

4. Results

To test the adequacy of this method in actual application, the final β functions obtained for first permanent ice and complete freeze-over were used to hindcast river ice formation at the site

under investigation during the 1964/65, 1966/67, 1969/70 and 1972/73 winter seasons. Using average departure-from-normal air temperatures with the computational procedure described in the previous example, forecasts were made beginning four weeks, three weeks, two weeks and one week prior to first permanent ice and complete freeze-over during the four winter seasons.

A comparison of the observed and computed dates of first permanent ice and complete freeze-over at Sault Ste. Marie appear in Table 4 and Table 5, respectively. The results for first permanent ice indicate that the computed dates generally occur within ± 3 days of the observed dates. On the average, the computed dates of first permanent ice occur within ± 1 day of the observed date. However, the results for complete freeze-over indicate that the computed dates infrequently occur within ± 3 days of the observed dates. It is interesting to note that, on the average, the computed dates of complete freeze-over occur 10 days after the observed date.

The computed dates of river ice formation having discrepancies larger than ± 3 days from the observed dates can be attributed to two major factors. First, the association of air temperature with ice formation limits the method to thermal processes only. The sensitivity of a water surface to variable wind conditions is one reason why observed dates of ice formation fluctuate within rather wide limits. The optimum condition for ice formation on a water surface is the presence of a weak wind, for an ice sheet may break up as a result of waves and currents produced by a

TABLE 3
Forecast of First Permanent Ice
Using Departure-From-Normal Air Temperatures

A	B	B'	C	D	E	F
Date	Daily Normal Air Temp (°C)	5.10°C Below Daily Normal Air Temp	(F - 1 Day)	(B' - C)	(D × 0.037)	Weighted Mean Temp (°C) (C + E)
November 22 Weighted Mean Temperature = 4.72						
November 23	-1.70	-6.80	4.72	-11.52	-.42	4.29
November 24	-1.70	-6.80	4.29	-11.09	-.41	3.88
November 25	-2.20	-7.30	3.88	-11.18	-.41	3.47
November 26	-2.20	-7.30	3.47	-10.77	-.40	3.07
November 27	-2.80	-7.90	3.07	-10.97	-.41	2.67
November 28	-2.80	-7.90	2.67	-10.57	-.39	2.27
November 29	-3.30	-8.40	2.27	-10.67	-.39	1.88
November 30	-3.30	-8.40	1.88	-10.28	-.38	1.50
December 1	-3.30	-8.40	1.50	-9.90	-.37	1.13
December 2	-3.90	-9.00	1.13	-10.13	-.37	.76
December 3	-3.90	-9.00	.76	-9.76	-.36	.40
December 4	-4.40	-9.50	.40	-9.90	-.37	.03
December 5	-4.40	-9.50	.03	-9.53	-.35	-.32

strong wind. However, strong northwest winds, characteristic of the winter season climate of the Sault Ste. Marie area, may push colder surface water from Whitefish Bay and eastern Lake Superior into the St. Marys River. The method of predicting river ice formation does not consider wind, and this advection of cold surface water may account for the fact that, on the average, the computed dates of complete freeze-over occur 10 days subsequent to the observed date. Second, an accumulation of snow cover on an initial ice sheet will depress the underlying ice, causing cracks and crevasses. To maintain a hydrostatic balance, water flows through the cracks in the ice, saturating the lowest layer of snow. Water may also saturate the entire snow layer, either through precipitation or by melting of the snow cover during a short period of warming. When the snow layer becomes super-saturated from surface melting, excess water percolates downward to the ice surface where a subsequent cooling may freeze the slob snow, causing a layer of snow-ice or white ice. This type of ice formation can occur several times during a winter season, thus producing highly variable ice coverage. The presence of white ice might be directly responsible for incorrect estimates of the dates of complete freeze-over on the St. Marys River between the Soo Locks and Gros Cap. Erroneous freeze-over dates are not uncommon in observer records, and such a misjudgement would obviously lead to an erroneous β function. Bilello (1964a) questioned an observation recorded at Quebec on the St. Lawrence River, and found an error of approximately four weeks!

5. Conclusion

From the results of this study, it appears that useful forecasts of river ice formation, computed objectively from past air temperatures, are practicable and can be done with a single variable. Although this method appears climatological, it differs from the standard approach in that a continually changing point of reference is provided, based on expected departure-from-normal air temperatures. If an accurate prediction of air temperature could be made thirty days in advance, then this method could be used as a long range tool. However, the greatest success is intended for extended range forecasts.

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TABLE 4
Comparison of Observed and Computed Dates of
First Permanent Ice at Sault Ste. Marie

Winter Season	Observed Date	Computed Dates			
		4 Week Fcst.	3 Week Fcst.	2 Week Fcst.	1 Week Fcst.
1964/65	06 December	06 December	03 December	05 December	05 December
1966/67	20 December	16 December	13 December	16 December	16 December
1969/70	15 December	13 December	14 December	16 December	15 December
1972/73	03 December	11 December	10 December	12 December	10 December
Average	11 December	11 December	10 December	12 December	11 December

TABLE 5
Comparison of Observed and Computed Dates of
Complete Freeze-Over at Sault Ste. Marie

Winter Season	Observed Date	Computed Dates			
		4 Week Fcst.	3 Week Fcst.	2 Week Fcst.	1 Week Fcst.
1964/65	13 January	28 January	30 January	29 January	23 January
1966/67	19 January	22 January	24 January	22 January	22 January
1969/70	10 January	19 January	18 January	21 January	19 January
1972/73	14 January	29 January	26 January	23 January	23 January
Average	14 January	24 January	24 January	24 January	22 January

REFERENCES

- Bilello Michael A. 1961: Formation, Growth and Decay of Sea Ice in the Canadian Arctic Archipelago. *Arctic*, 14, pp 1-24.
- Bilello, Michael A., 1964a: *Ice Prediction Curves for Lake and River Locations in Canada*, Research Report 129, U.S. Army Cold Regions Research and Engineering Laboratory, Hanover, New Hampshire, 12 pp.
- Bilello, Michael A., 1964b: Method for Predicting River and Lake Ice Formation. *Journal of Applied Meteorology*, 3, pp 38-44.
- Canadian Meteorological Service 1971: *Freeze-up and Break-up Dates of Dates of Water Bodies in Canada*, CLI-1-71, Toronto, 75 pp.
- Noble Vincent E. 1965: Winter Temperature Structure of Lake Michigan. *Proceedings of Eighth Conference on Great Lakes Research*, Ann Arbor, Michigan, pp. 334-341.
- Poulin, R. Y., J. R. Robinson and D. F. Witherspoon 1971: Probability Forecasts of the St. Lawrence River Between Kingston, Ontario and Sorel, Quebec. *Proceedings of Twenty-eighth Eastern Snow Conference*, Fredericton, N. B., Canada, pp. 33-48.
- Richards, T. L., 1964: The Meteorological Aspects of Ice Cover on the Great Lakes. *Monthly Weather Review*, 92, pp. 297-302.
- Rodhe, Bertile 1952: On the Relationship Between Air Temperature and Ice Formation in the Baltic. *Geografiske Annaler*, 34, pp. 141-163.
- Rodhe, Bertile, 1955: A Study of the Ice Extent, the Course of Air Temperature and the Sea Surface Temperature in the Aland Archipelago. *Geografiska Annaler*, 35, pp. 175-202.
- Shuliakovskii, L. G., ed., 1966: *Manual of Ice-Formation Forecasting for Rivers and Inland Lakes*, trans. by Israel Program for Scientific Translation, Jerusalem, 245 pp.
- Williams, G. P., 1965: Correlating Freeze-up and Break-up with Weather Conditions. *Canadian Geotechnical Journal*, 2, pp. 313-326.

APPENDIX

As developed by Rhode (1952), the theory of a weighted-mean air temperature is based upon Newton's law of cooling. This law states that the rate of heat loss from a water body is proportional to the difference between the temperature of the body and that of the ambient medium. Applying this law to the heat trans-

fer between the atmospheric boundary layer and a water body, we represent the rate of upward heat transfer (q) by the formula;

$$q = -a(T - \tau) \quad (1)$$

where T is the air temperature, τ is the water-surface temperature and a is constant. Assuming this heat is transferred from a subsurface layer of depth h and the mean temperature of the layer is lowered γ times as much as the surface temperature ($0 < \gamma < 1$), we derive the relation:

$$q = -c\gamma h \rho \delta\tau / \delta t \quad (2)$$

where c is the specific heat of water, ρ is the density of water and t is time.

Combining equations (1) and (2) we receive:

$$\delta\tau / \delta t = k(T - \tau) \quad (3)$$

where $k = a / c\gamma h \rho$ and will be treated as a constant with the inverse dimension of time. In order to solve this linear differential equation of the first order, we multiply by e^{kt} :

$$e^{kt}\delta\tau + k\tau e^{kt}dt = kTe^{kt}dt \quad (4)$$

Integrating from an initial time t_0 to some specified later time t^n , we obtain:

$$\tau_n e^{kt_n} - \tau_0 e^{kt_0} = \int_{t_0}^{t_n} T \delta(e^{kt}) \quad (5)$$

where $\tau_0 e^{kt_0}$ is a constant specified by the lower boundary conditions of integration.

We may divide the period between t_0 and t_n into equal intervals so that $t_1 - t_0 = t_2 - t_1 = t_3 - t_2 = \dots = t_n - t_{n-1} = \Delta t$. The intervals are chosen in this manner to permit a substitution of the air temperature by its mean value. Letting T_ν represent the mean during the interval from $t_{\nu-1}$ to t_ν , we obtain:

$$\tau_n e^{kt_n} - \tau_0 e^{kt_0} = \sum_{\nu=1}^n T_\nu \int_{t_{\nu-1}}^{t_\nu} \delta(e^{kt}) \quad (6)$$

and solving for τ_n becomes:

$$\tau_n = \tau_0 e^{-k(t_n - t_0)} + (1 - e^{-kt}) \sum_{\nu=1}^n T_\nu e^{-k(t_n - t_{\nu-1})} \quad (7)$$

Writing:

$$(1 - e^{-k\Delta t}) = [1 - e^{-k(t_n - t_0)}] [1 + e^{-k(t_n - t_{n-1})} + \dots + e^{-k(t_n - t_1)}]^{-1} \quad (8)$$

we obtain:

$$\tau_n = \tau_0 e^{-k(t_n - t_0)} + [1 - e^{-k(t_n - t_0)}] \xi_n \quad (9)$$

where:

$$\xi_n = \left[\sum_{\nu=1}^n T_\nu e^{-k(t_n - t_{\nu-1})} \right] \left[\sum_{\nu=1}^n e^{-k(t_n - t_{\nu-1})} \right]^{-1} \quad (10)$$

is the weighted mean of air temperatures during the period t_0 to t_n . The further in the past an air temperature is read into the equation, the less is its weight by the coefficient k^{-1} . It follows, that as the period from t_0 to t_n is increased, the influence of the condition τ_0 is decreased. Therefore, if the initial time is chosen as a point far enough in the past, the lower boundary condition becomes unimportant. Simultaneously, the term $1 - e^{-k(t_n - t_0)}$ approaches unity, and τ_n becomes essentially equal to the weighted mean ξ_n of all past air temperatures. Originally, τ was defined as water-surface temperature but can now be treated as a function of air temperature alone. Therefore, according to equation (10), we may expect the formation of ice on a water surface when:

$$\xi_n = [1 - e^{-k\Delta t}] [1 - e^{-(n+1)k\Delta t}]^{-1} \cdot [T_n + T_{n-1} e^{-k\Delta t} + \dots + T_1 e^{-(n-1)k\Delta t}] = 0 \quad (11)$$

In this equation, the formation of ice is established by the coefficient k^{-1} , and the progression of air temperature is represented by the time sequent T_1, T_2, \dots, T_n .

A more operational approach to the calculation of τ may be established by utilizing a formula of iteration. Replacing t_0 with t_{n-1} as the lower boundary condition in equation (7) we receive:

$$\tau_n = \tau_{n-1} e^{-k\Delta t} + (1 - e^{-k\Delta t}) T_n \quad (12a)$$

or

$$\tau_n = \tau_{n-1} + (1 - e^{-k\Delta t}) (T_n - \tau_{n-1}) \quad (12b)$$