

WHAT DOES QUASI-GEOSTROPHIC REALLY MEAN?

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Author's note: *The first in a series of continuing education articles on Quasi-Geostrophic (QG) theory presented here may seem a bit academic, rather than operationally oriented. These fundamentals, however, are necessary to set the stage for more operationally useful material that will follow in future articles.*

1. Introduction and Motivation

Quasi-geostrophic theory (see Holton 1992 or Bluestein 1992 for a thorough review of this subject) has been the backbone of synoptic-scale weather forecasting in the middle latitudes for decades (Sanders and Hoskins 1990). It's simplicity has allowed weather forecasters to successfully diagnose large-scale vertical motions and height tendencies that result from easily understood physical concepts (Durran and Snellman 1987). Diagnostics such as mid-tropospheric vorticity advection and low-level temperature advection, vorticity advection by the thermal wind (known to some as PIVA/NIVA, Trenberth 1978), and divergence patterns of Q-vectors (Hoskins et al. 1978; Hoskins and Pedder 1980) have all been used effectively in searching for areas of large-scale vertical motions. In turn, these vertical motion patterns have provided insight into familiar conceptual models such as the secondary circulations around the entrance and exit regions of jet streaks and near frontal systems.

Three common charts used to diagnose patterns of vertical velocities include advection of vorticity at 500 mb (Fig. 1a), advection of vorticity by the thermal wind (Fig. 1b), and divergence of Q-vectors (Fig. 1c). These tools should be very familiar to forecasters and most can glance at the diagrams and quickly locate the areas of concern. What is not well-known (or remembered) is that all three methods originate from the same QG approximation. Additionally, forecasters tend to forget or overlook the basis and assumptions that define the QG approximation and may even employ these QG tools inappropriately.

Part of this lack of understanding can be explained in historical terms. In the days of facsimile or remotely produced weather charts, it was difficult to implement or utilize various QG tools effectively (Dunn 1991). Forecasters were limited to simple graphical signatures such as 500-mb vorticity advection and advection of 500-mb vorticity by the 1000-500 mb thickness. Through repetition of these limited techniques, weather prediction became more of a "pattern-matching" exercise versus a true scientific analysis of what the numerical models were trying to depict. In this framework, it was easy to forget the basic underlying assumptions and other applications of QG theory.

With the advent of economical computer power and the availability of gridded data at many forecast offices, QG theory and its associated tools have experienced something of a rebirth. Values of QG forcing such as positive advection of geostrophic vorticity by the geostrophic wind and Q-vector convergence can be calculated and displayed in a matter of seconds on both personal computers and higher-powered scientific workstations.

Forecasters have been and are currently gaining the ability to calculate and display not only the various QG terms, but virtually any meteorological field. Cheaper computing power is also affecting the grids received at forecast offices as the operational numerical weather prediction models trend downward toward the mesoscale.

With these increased capabilities comes the responsibility to keep current in the science and to "rediscover" the wealth of information previously mastered in academic and professional careers. This wealth includes a remembrance of the limitations as well as the various applications of QG theory. Forecasters need to understand what to expect from QG theory and to learn when it may not explain all of the forcings in an area of interest.

It is also important to realize that some operational models are approaching the scale where QG assumptions are less valid (e.g., 29 Km and 10 Km versions of the mesoscale Eta model, see Black 1994 for a description of the mesoscale Eta model). To this end, forecasters need to know WHEN NOT to use the QG approximation. Synoptic-scale diagnosis will remain important, however, since mesoscale models tend to error significantly when errors in the large-scale forecast can be identified. Additionally, mesoscale models typically use synoptic-scale models for boundary conditions, and errors in these boundary conditions can lead to significant errors on the mesoscale.

With these thoughts in mind, a series of continuing education articles has been developed. Unlike many academic experiences, the articles focus on the physical meaning of various QG concepts rather than dwell on mathematical derivation. Operational examples and usage are also emphasized. Unless stated otherwise, it is assumed that the articles are applicable mainly to the mid-latitudes. The series does not propose that QG tools are the best or only way to diagnose the synoptic-scale variables. The historical usage and simplicity of these methods does, however, make QG-related topics a natural place to commence a continuing education series.

The articles deal with several topics including: the meaning of QG, examples of QG approximation, the traditional omega equation, the Trenberth (1978) formulation, the Q-vector formulation (Hoskins et al. 1978), and an explanation of Q-vectors (Hoskins and Pedder 1980). It is hoped that these articles will bring the reader up-to-date with QG-related tools and assumptions, and better prepare him/her for rapid changes that continue to occur in operational forecasting. This first installment deals with the question: What does quasi-geostrophic really mean?

2. Starting Point—Basic Momentum Equations

In order to derive the meaning of quasi-geostrophic, it is important to first understand the simplest balance state in the atmosphere—geostrophic balance. Recall the terms from the horizontal momentum equations (in isobaric coordinates):

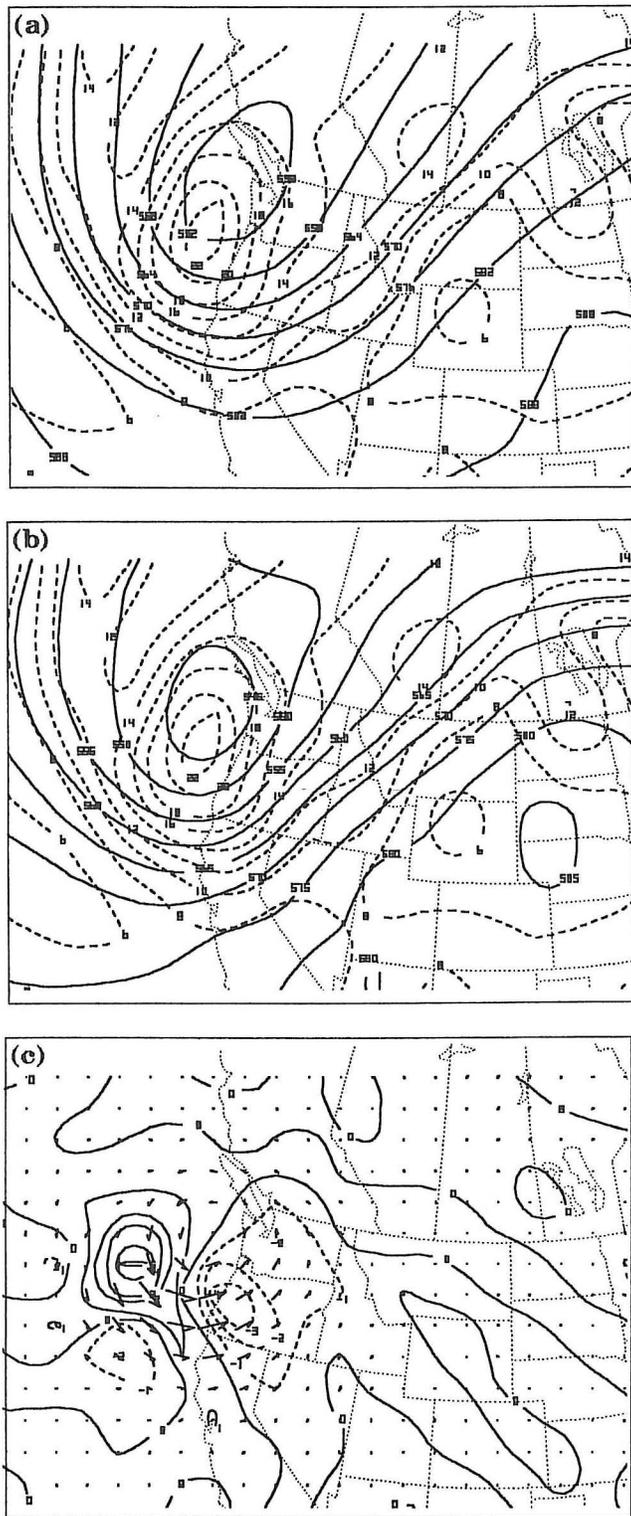
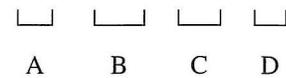


Fig. 1. Three charts to infer patterns of vertical motion in the mid-troposphere valid 0000 UTC 17 August 1995: (a) shows geopotential heights (solid, 60-m interval) and absolute vorticity (dashed, $2 \times 10^{-5} \text{ s}^{-1}$ interval) at 500 mb from which positive or negative vorticity advection can be estimated, (b) displays 1000-500 mb thickness (solid, 50-m interval) and absolute vorticity at 500 mb (dashed, $2 \times 10^{-5} \text{ s}^{-1}$ interval) from which advection of the vorticity by the thermal wind can be deduced, and (c) shows the divergence of Q-vectors at 500 mb where convergence (dashed) implies upward motion and divergence (solid) implies downward motion (in units of $\text{K m}^{-2} \text{ s}^{-1}$, not scaled by static stability parameter).

$$\frac{Du}{Dt} = fv - \frac{\partial\phi}{\partial x} + F_x \quad (1)$$

$$\frac{Dv}{Dt} = -fu - \frac{\partial\phi}{\partial y} + F_y \quad (2)$$



These equations are in the same form as Newton's Second Law which says that the acceleration of a parcel equals the sum of the forces (per unit mass) acting on that parcel. (All terms are in the form of accelerations, but terms on the right hand side can be thought of as forces as long as it is remembered that these terms are actually forces per unit mass.) Thus, equations (1) and (2) state "the acceleration of an air parcel (left hand side—LHS) is equal to the sum of the accelerations on the right hand side (RHS). The terms on the RHS can be thought of as the physical mechanisms which produce a parcel's acceleration. In short, the terms represent:

- A = acceleration of a parcel of air
- B = coriolis acceleration
- C = height gradient acceleration (like pressure gradient acceleration)
- D = frictional acceleration

Both the height gradient and frictional terms are fairly easy to understand. The coriolis term results from a coordinate system that is fixed to the rotating earth. Remember that an object is accelerating unless it is motionless or moving with a constant velocity with respect to a point in space. If one is standing on the earth, he/she is accelerating simply due to the earth's rotation. This acceleration is typically taken into account in a term dealing with the gravitational acceleration. An additional term, the coriolis acceleration, is necessary if this person is moving on the surface of the earth.

Equations (1) and (2) above are the Lagrangian forms of the momentum equations, i.e., moving along with a parcel of air or following the airflow. For the change of wind with time at a point (as required in most meteorological applications), the Eulerian form is required. The LHS of (1) and (2) may be rewritten:

$$\frac{Du}{Dt} = \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + \omega \frac{\partial u}{\partial p} \quad (3)$$

$$\frac{Dv}{Dt} = \frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} + \omega \frac{\partial v}{\partial p} \quad (4)$$



These equations simply show that the acceleration of a parcel is due to the change in the wind with time at a point (A) and the non-linear advection terms (B). The advection terms are not as easily understood because of the non-linearity (multiplying wind components by wind components), but it might be thought of as sort of a "wind blowing the wind along". Combining (1) and (3) as well as (2) and (4) gives the Eulerian form:

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + \omega \frac{\partial u}{\partial p} = fv - \frac{\partial\phi}{\partial x} + F_x \quad (5)$$

$$\frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} + \omega \frac{\partial v}{\partial p} = -fu - \frac{\partial\phi}{\partial y} + F_y \quad (6)$$

which gives (with minor rearrangement) the change in wind with time at a point.

3. The Simplest Balance

It's no secret that the atmosphere tries to remain in a balanced state. In fact, above the earth's surface (well away from the boundary layer), the atmosphere is always nearly in balance—geostrophic balance. Assuming frictionless flow and no acceleration of a parcel (LHS = 0), (1) and (2) become:

$$fv_g = \frac{\partial\phi}{\partial x} \quad fu_g = -\frac{\partial\phi}{\partial y} \quad (7)$$

These equations simply show that the coriolis acceleration is balanced by the height gradient acceleration (Fig. 2)—the definition of geostrophic balance. The u_g and v_g terms are the x and y components of the geostrophic wind, respectively. A more rigorous scale analysis would show that the terms in equation (7) are the most dominant (largest) at synoptic scales.

Geostrophy is nice and simple, but nothing "exciting" occurs in the atmosphere under geostrophic balance. For the most part, the geostrophic wind is nearly non-divergent or can be defined as non-divergent (see Holton, 1992, for definition of a non-divergent geostrophic wind), and divergence is necessary to produce the large-scale vertical motions of interest. In the QG approximation, an ageostrophic term is retained to provide the required divergence and parcel accelerations.

4. Quasi-Geostrophic Approximation

Using the fact that the total wind can be divided into geostrophic and ageostrophic parts ($u = u_g + u_a, v = v_g + v_a$), equations (5) and (6) can be written by substitution:

$$\begin{aligned} &\frac{\partial(u_g + u_a)}{\partial t} + (u_g + u_a) \frac{\partial(u_g + u_a)}{\partial x} \\ &+ (v_g + v_a) \frac{\partial(u_g + u_a)}{\partial y} + \omega \frac{\partial(u_g + u_a)}{\partial p} \\ &= f(v_g + v_a) - \frac{\partial\phi}{\partial x} + F_x \end{aligned} \quad (8)$$

$$\begin{aligned} &\frac{\partial(v_g + v_a)}{\partial t} + (u_g + u_a) \frac{\partial(v_g + v_a)}{\partial x} \\ &+ (v_g + v_a) \frac{\partial(v_g + v_a)}{\partial y} + \omega \frac{\partial(v_g + v_a)}{\partial p} \\ &= -f(u_g + u_a) - \frac{\partial\phi}{\partial y} + F_y \end{aligned} \quad (9)$$

Somewhere in here lies the QG version of a horizontal momentum equation. It turns out (by scale analysis) that all ageostrophic terms can be neglected on the LHS of both (8) and (9) along with the vertical advection terms (last term on LHS). These terms are small compared to the remaining (geostrophic) terms on the LHS. Frictionless flow can be assumed if the approximations are used well away from the boundary layer. The resulting equations are shown below:

$$\frac{\partial u_g}{\partial t} + u_g \frac{\partial u_g}{\partial x} + v_g \frac{\partial u_g}{\partial y} = f(v_g + v_a) - \frac{\partial\phi}{\partial x} \quad (10)$$

$$\frac{\partial v_g}{\partial t} + u_g \frac{\partial v_g}{\partial x} + v_g \frac{\partial v_g}{\partial y} = -f(u_g + u_a) - \frac{\partial\phi}{\partial y} \quad (11)$$

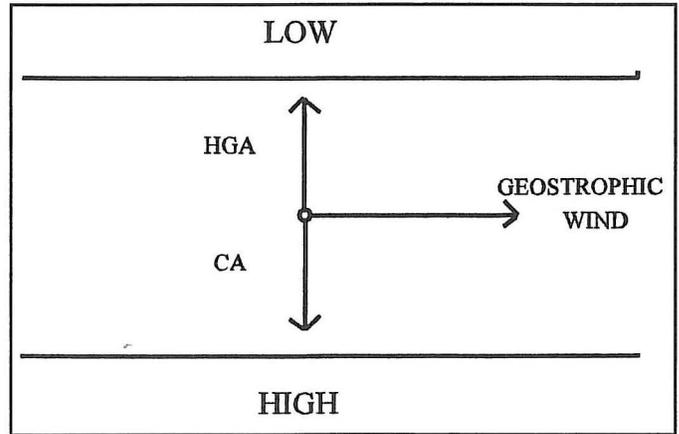


Fig. 2. Conceptual diagram of geostrophic balance. Horizontal lines are height contours with heights decreasing toward the top of the diagram. Vertical arrows represent the two balancing accelerations, coriolis (CA) and height gradient (HGA, similar to pressure gradient). Geostrophic wind is horizontal vector with direction determined by orientation of height contours and speed by the contour packing.

Note that only geostrophic terms remain on the LHS of (10) and (11). The natural question is "Why can the ageostrophic terms on the LHS be neglected while the ageostrophic coriolis terms (fv_a or fu_a) on the RHS are retained?". It turns out that the fv_a and fu_a terms on the RHS are of the same order of magnitude (roughly 10^{-4}) as the retained geostrophic terms on the LHS. Besides, as will be evident later, if the ageostrophic terms on the RHS are neglected too, the equation once again only represents geostrophic balance!

Equations (10) and (11) can be written in Lagrangian form (parcel form) with some minor rearrangement on the RHS:

$$\frac{D_g u_g}{Dt} = fv_g - \frac{\partial\phi}{\partial x} + fv_a \quad (12)$$

$$\frac{D_g v_g}{Dt} = -fu_g - \frac{\partial\phi}{\partial y} - fu_a \quad (13)$$

The terms have been drawn in a size relative to their magnitudes. The larger two "middle" terms in both equations are simply the terms involved in the geostrophic wind balance (eq. 7) and, thus, cancel one another. The "outside" or smaller terms, which must be equal to each other, represent the required parcel acceleration that was the object of the above manipulations. The LHS terms can be thought of as the rate of change of the geostrophic wind in a parcel moving with the geostrophic flow while the RHS small terms can be thought of as the "forcing" mechanism which produces this acceleration.

Figure 3 displays representative magnitudes of the coriolis terms stated above in the x-component momentum equation (eq. 12): fv_g and fv_a . Since fv_g is equal to $\partial\phi/\partial x$ by the definition of the geostrophic wind, the solid lines are representative of the east-west height gradient. The solid line couplet centered just off the Pacific Northwest coast depicts a mid-troposphere trough. Comparing the solid (fv_g) and dashed (fv_a) lines, it is evident that the geostrophic term (fv_g) is dominant as stated in (12). The same can be shown for the y-component QG horizontal momentum equation (13) using fu_g and fu_a .

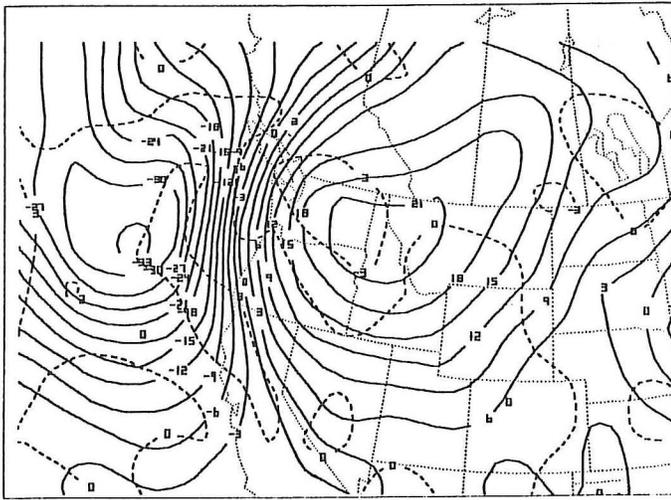


Fig. 3. Representative magnitudes of the coriolis terms in the quasi-geostrophic approximation of the x-component horizontal momentum equation showing that the geostrophic term, fv_g (solid), is typically larger than the ageostrophic term, fv_a (dashed). Both are in units of $3 \times 10^{-4} \text{ m s}^{-2}$.

Since the two large terms in both (12) and (13) cancel one another, the atmosphere is dominated by geostrophic balance. In other words, the atmosphere is nearly in geostrophic balance or QUASI-GEOSTROPHIC. It is the small, but significant ageostrophic terms on the RHS of (12) and (13) which provide the “interesting weather” and which lead to evolving changes in the geostrophic flow.

This rather simple discussion, based mainly on the horizontal momentum equation, should give the reader at least a cursory understanding of the meaning of the term quasi-geostrophic. A complete QG set includes QG forms of the horizontal momentum, continuity, thermodynamic, and hydrostatic equations. These equations form a closed set, i.e., there are an equal number of dependent variables and equations. From this set, other interesting prognostic and diagnostic equations can be derived such as the QG form of the vorticity equation.

5. Quasi-Geostrophic Vorticity

The vertical component of the vorticity of the wind is a familiar concept to operational forecasters (Fig. 1a). The absolute vorticity is made up of two parts: relative vorticity (ζ) or that which is moving along with the rotating coordinate system on earth and the earth’s vorticity (f) due to the earth’s rotation. The vertical component of the relative vorticity can be defined as:

$$\zeta = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \tag{14}$$

Simply stated, this equation shows the combined effects of how the north-south wind varies in the east-west direction and how the east-west wind varies in the north-south direction. In other words, relative vorticity represents a point measure of the rotation of the airflow moving along with the rotating coordinate system on the earth. The QG form of the vertical component of relative vorticity can be found by simply substituting the geostrophic wind for the actual wind:

$$\zeta_g = \frac{\partial v_g}{\partial x} - \frac{\partial u_g}{\partial y} \tag{15}$$

In operational meteorology, the *tendency*, or how this vorticity changes with time, is important. To obtain the QG vorticity equation, (12) and (13) can be manipulated to yield:

$$\frac{\partial \zeta_g}{\partial t} = \underbrace{-\vec{V}_g \cdot \nabla(\zeta_g + f)}_A + \underbrace{f_0}_{B} \underbrace{\frac{\partial \omega}{\partial p}}_C \tag{16}$$

The equation may appear a bit intimidating to some in its vector form, but its physical simplicity is the important point here. (f_0 denotes that the coriolis parameter is held constant.) The individual terms are:

- term A—change of geostrophic relative vorticity with time at a point
- term B—advection of absolute geostrophic vorticity by the geostrophic wind
- term C—stretching/compressing in a vertical column of air

Simply put, (16) states that geostrophic relative vorticity can be changed at a point by blowing vorticity around (advecting it) and/or generating/dissipating vorticity by stretching/compressing the column of air over the point in question. It is clear that advecting vorticity can change its value at a point with time. What about the divergence or stretching term?

First, stretching is directly related to convergence by the continuity equation:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = -\frac{\partial \omega}{\partial p} \tag{17}$$

so that when the vertical column is stretched, horizontal convergence tends to occur. Alternatively, horizontal divergence is associated with compression of the vertical column. The process can also be illustrated by visualizing a parcel whose volume remains constant but whose vertical and horizontal dimensions are allowed to change. If the parcel is stretched, then it will contract horizontally (converge). When it is compressed, it will expand horizontally (diverge).

Figure 4a displays a common conceptual model of the vertical distribution of omega in the troposphere. This distribution implies a stretching of the column in the lower half and com-

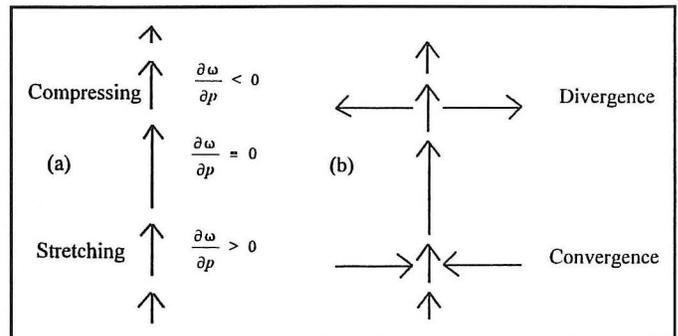


Fig. 4. Conceptual model of airflow through vertical column in atmosphere. Vertical arrows give sense of upward/downward motion. (a) shows stretching in lower levels and compression above while (b) displays corresponding convergence and divergence pattern.

pressing of the column in the upper half. Alternatively, Fig. 4b shows the vertical distribution of divergence associated with Fig. 4a. If advection is ignored, (16) can be rewritten as:

$$\frac{\partial \zeta_g}{\partial t} = f_0 \frac{\partial \omega}{\partial p} \quad (18)$$

Thus, geostrophic relative vorticity is tending to decrease with time (aloft in Fig. 4) given either compression or divergence and increase (lower levels in Fig. 4) given either stretching or convergence. Returning to the parcel illustration, as the parcel expands vertically, air converges towards the axis of rotation causing it to rotate faster. When the parcel compresses vertically, air diverges from the axis of rotation causing the parcel to spin more slowly.

At first glance, this relationship in (18) may appear to be at odds with experience. Note that in Fig. 5, the area downstream from the upper trough at 300 mb is generally described by divergence of the wind field. Inspection of this same trough in Fig. 1a shows that vorticity should be increasing with time downstream from the trough axis. Equation (18) states that geostrophic relative vorticity should be decreasing with time in this area due to the divergence or compression. Obviously, the advection term which was ignored in (18), but shown in the full equation in (16), is the dominant term in the upper levels. In essence, despite a decreasing trend in geostrophic relative vorticity due to the vertical motion pattern, the fast wind speeds and high values of vorticity aloft produce a more dominant advection pattern which increases the geostrophic vorticity downstream of the trough with time. Simply stated, upper-level vorticity advection usually dominates changes in vorticity due to the divergence aloft. In lower levels (not

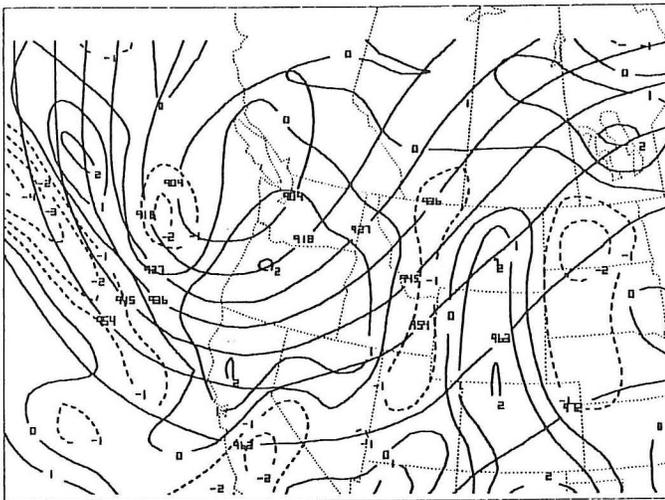


Fig. 5. Divergence of the wind field at 300 mb valid 0000 UTC 17 August 1995. Divergence is solid and convergence is dashed in units of 10^{-5} s^{-1} .

shown), wind speeds are lighter and vorticity tends to be aligned with the height contours (channeled) so that the convergence pattern is more likely to be important. This conceptual pattern will be important when changes in vorticity advection with height are investigated in future articles.

The point of the above discussion is to demonstrate the simplicity of the QG framework using the QG vorticity equation. This equation can be compared to the less simplified form (see Holton, chapter 4) involving other less obvious terms such as tilting or twisting and the solenoidal effect. In summary, to a first approximation, the QG form of the vertical component of vorticity is easily understood and describes the synoptic-scale tendency of vorticity rather well. This simplicity will be evident in other more useful equations in future installments.

6. Summary

In a simple and rather non-rigorous fashion, the meaning of the term quasi-geostrophic has been demonstrated using the horizontal momentum equations. It was shown that it was necessary to retain a small ageostrophic term in these equations in order to produce “interesting weather” in the synoptic-scale atmosphere. These simplified equations will be necessary to derive the height-tendency and various forms of the omega equation.

A simplified QG vorticity tendency equation was also discussed. The purpose was not to confuse the reader with detailed mathematics, but to show that the terms in this equation are easily understood. More complex equations will be displayed in future articles, but the emphasis will be the same—understanding the physical nature of the terms.

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