REVIEW OF QG THEORY—PART II
THE OMEGA EQUATION

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Editor's note: *This is the second in a series of continuing education articles on QG theory. The first "What Does Quasi-Geostrophic Really Mean?", was published in the* National Weather Digest *Volume 21 Number 1. David Billingsley, Science and Operations Officer, NOAA/NWSFO, Boise, Idaho, volunteered to author this timely series and his initiative, time and special efforts are deeply appreciated. Supportive comments have been received from many readers.*

1. Introduction and Motivation

Diagnoses of the synoptic-scale vertical motion field (omega) are an integral part of the forecasting process. Most notably, the degree and longevity of large-scale lift affects the stability and moisture distribution of the atmosphere. For example, sustained lift may steepen the environmental lapse rates to "prepare" the atmosphere for a convective event, or it may produce areas of significant large-scale condensation and precipitation in advance of a mid-tropospheric disturbance. Tools used "in search of omega" range from post-processed omega fields calculated directly from the numerical weather prediction model output, advection and evolution of features on satellite imagery (including trends in the water vapor imagery), estimation or calculation of lift produced by isentropic flow, correlation of the structure of a conceptual model of an atmospheric disturbance and lift, estimation or calculation of the quasi-geostrophic (QG) forcing functions from the omega equation, to the direct calculation of QG omega.

This article will focus on the QG omega equation and its utility in diagnosing large-scale vertical motions. In its traditional form, the omega equation is as follows:

$$\underbrace{\left(\nabla_h^2 + \frac{f_0^2}{\sigma}\frac{\partial^2}{\partial p^2}\right)\omega}_{A} = \underbrace{-\frac{f_0}{\sigma}\frac{\partial}{\partial p}[-\vec{V}_g \cdot \nabla(\zeta_g + f)]}_{B} \underbrace{- \frac{1}{\sigma}\nabla^2\left[-\vec{V}_g \cdot \nabla\left(-\frac{\partial\phi}{\partial p}\right)\right]}_{C} \quad (1)$$

Though the equation may appear a bit intimidating to those who have been away from academic experience for some time, it is certain that most operational meteorologists are intimately familiar with its terms. For instance, term B is proportional to how vorticity advection changes with height and term C is proportional to the temperature advection.

The derivation of the omega equation involves manipulation of the QG vorticity equation, the QG thermodynamic equation, and a few others (see Holton 1992 or Bluestein 1992 for details). It is rarely used in its full form as shown in (1) above. The equation can be solved explicitly for omega, but the solution is not trivial. Forecasters generally make assumptions about the structure of the atmosphere to further simplify the equation so that omega can be more easily related to the forcing functions on the right hand side. It is in this light that the equation will be examined in this article.

Why is it important to review this well-known equation? For several reasons including the following:

1) Though more exact equations are used in the modern numerical weather prediction models to produce a vertical motion field, it is advantageous for the forecaster to be able to relate some physical mechanism to these vertical motion patterns. In other words, why does the vertical velocity field look the way it does? The omega equation, and in particular, the forcing functions of the right hand side provide such a mechanism.
2) The assumptions used in qualitatively estimating omega and in applying or simplifying its forcing functions tend to be forgotten or overlooked, especially the longer a forecaster has been away from academia. In fact, some operational meteorologists may not remember ANY connection between the forcing functions and QG theory.
3) With the advent of gridded numerical model data and more powerful computing tools in the operational environment, forecasters can, for the most part, produce any field they wish. For instance, positive vorticity advection in the mid-troposphere can be calculated instead of qualitatively estimated by the intersection of lines on a chart. Even better, differential vorticity advection or the Laplacian of temperature advection can be calculated and even combined. Use of the forcing functions in these ways necessitates better understanding of their origin, usefulness, and limitations.
4) Different derivations of the omega equation (Trenberth 1978; Hoskins et al. 1978) can be understood in light of the traditional omega equation. (These will be discussed in future articles.)
5) With the improvement of numerical models and the trend toward the mesoscale, there is a growing debate on the necessity, appropriateness, and usefulness of diagnosis methods based on QG theory. Since the forecaster ultimately has to make the decision on forecast methodology, it behooves her/him to know as much as possible about these methods and the theory from which they are derived.

From the above discussion, it is apparent that QG theory and the omega equation remain an important topic in operational meteorology. Even with the push to finer scale models, forecasters will still want to extract synoptic-scale information from these datasets. In an approach such as the forecast funnel (Snellman 1969), it is critical to assess the large-scale environment in order to understand the context in which the mesoscale processes are occurring. Though not the only way to view this large-scale picture, QG theory and its tools certainly provide a rather simple and understandable model.

2. Term-by-Term Examination

In a simplified form, the traditional omega (ω) equation says that synoptic-scale vertical motion, represented by term A in equation (1), is proportional to the sum of the differential vorticity advection (term B), and the Laplacian of thickness advection (term C). In essence, this equation is the basis for the use of vorticity and temperature advection as tools for diagnosing large-scale lift. What follows is a brief explanation of the individual terms.

a. Three-dimensional Laplacian of omega

Term A in equation (1) represents the three-dimensional Laplacian of omega; not exactly a common function used in forecasting the weather. It turns out that if wavelike behavior of omega (both horizontally and vertically) can be assumed, the three-dimensional Laplacian of omega can be shown to be proportional to minus omega:

$$\left(\nabla_h^2 + \frac{f_0^2}{\sigma}\frac{\partial^2}{\partial p^2}\right)\omega \propto -\omega \tag{2}$$

which when combined with equation (1) yields:

$$-\omega \propto \textit{forcing functions } B + C \tag{3}$$

This assumption is critical since it allows the forecaster to qualitatively relate the forcing functions to omega without complex mathematical manipulations involving the inversion of a Laplacian. Since minus omega is proportional itself to upward motion, statement (3) says that if the right hand side of the omega equation is positive, upward vertical motion can be assumed. Conversely, if the right hand side is negative, downward motion can be assumed. In essence, this means positive values of the forcing functions are associated with upward motion and negative values signify downward motion.

Forecasters should have little trouble convincing themselves that the atmosphere is typically wavelike, especially when viewed from a synoptic-scale framework. An example of simple vorticity advection from the ETA model valid at 1800 UTC 22 July 1996 displays this wavelike behavior (Fig. 1). It is easy to imagine that the resulting QG omega field forced by the "waves" in Fig. 1 would also be wavelike, assuming vorticity advection at this level is the predominant forcing mechanism. Note that some of the waviness across the central plains may have been induced by model terrain effects (Barnes et al. 1996). QG omega usually exhibits wavelike behavior in the vertical as well. A plot of omega versus pressure downstream of a mid-tropospheric disturbance (Fig. 2) typically yields a mid-tropospheric minimum in omega (maximum in upward motion). Though only "half" of a wave in the vertical, this pattern is still considered wavelike.

Being wavelike is an easy concept to grasp. On the other hand, the Laplacian of some variable is not quite so straightforward. It is helpful to understand the Laplacian, though, since it reappears from time to time in the QG framework. For instance, geostrophic vorticity can be shown to be proportional to the Laplacian of the height field. The Laplacian also surfaces in term C of the omega equation and in the QG height tendency equation (not discussed in this series). Hence, grasping this concept will help the reader with more than just the interpretation of the omega equation.

What is the Laplacian of omega? By simply looking at equation (1), this term can be defined as the sum of the forcing functions B and C (differential vorticity and temperature advection terms). Thus, the Laplacian of omega IS the QG forcing since it is equal to the right hand side of equation (1). The question to answer is, "What is the relationship between this forcing and the resulting omega field?"

This question has been partially answered already since it has been stated that given a wavelike field of omega, the Laplacian of omega is proportional to minus omega (statement (2)). This point can be easily illustrated by examination of these terms in one-dimension using a simple sine wave to represent omega (Fig. 3a). This wave is simply an omega couplet with descent on the left (positive omega) and ascent on the right (negative omega), similar to what might be found in a mid-tropospheric disturbance. The Laplacian of omega in one-dimension can be written as follows:

$$\nabla_x^2\omega = \frac{\partial}{\partial x}\left(\frac{\partial \omega}{\partial x}\right) \tag{4}$$

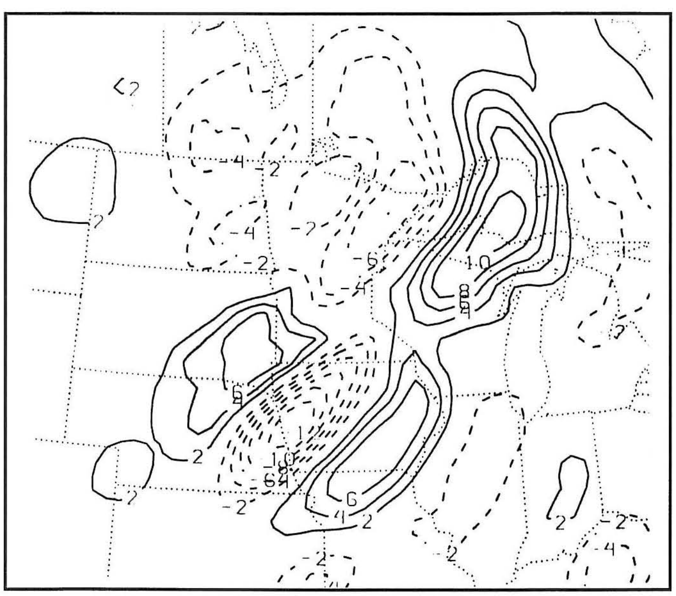

Fig. 1. 500-mb absolute vorticity advection from NWS/NCEP Eta model valid at 1800 UTC 22 July 1996. Solid (dashed) contours represent positive (negative) values. Contour interval is $2 \times 10^{-9}\,s^{-2}$.

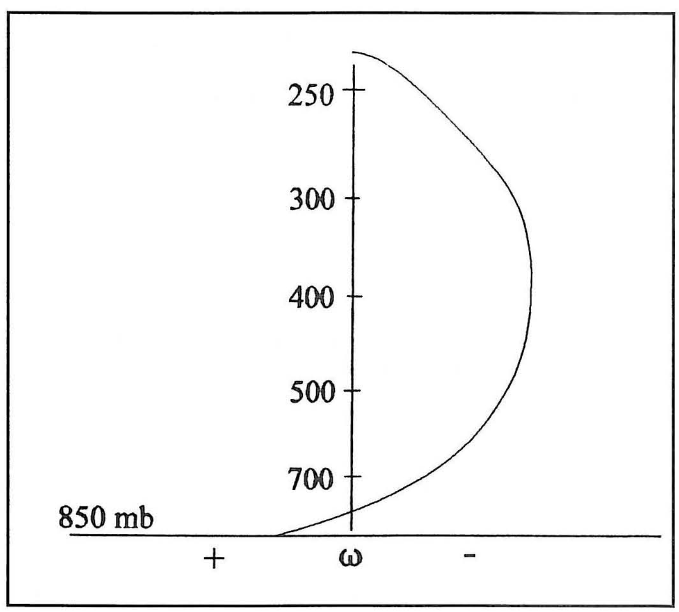

Fig. 2. Conceptual plot of omega versus pressure downstream of a typical mid-tropospheric disturbance. Maximum in upward motion occurs around 400 mb. (Note: author lives in elevated terrain of the west!)

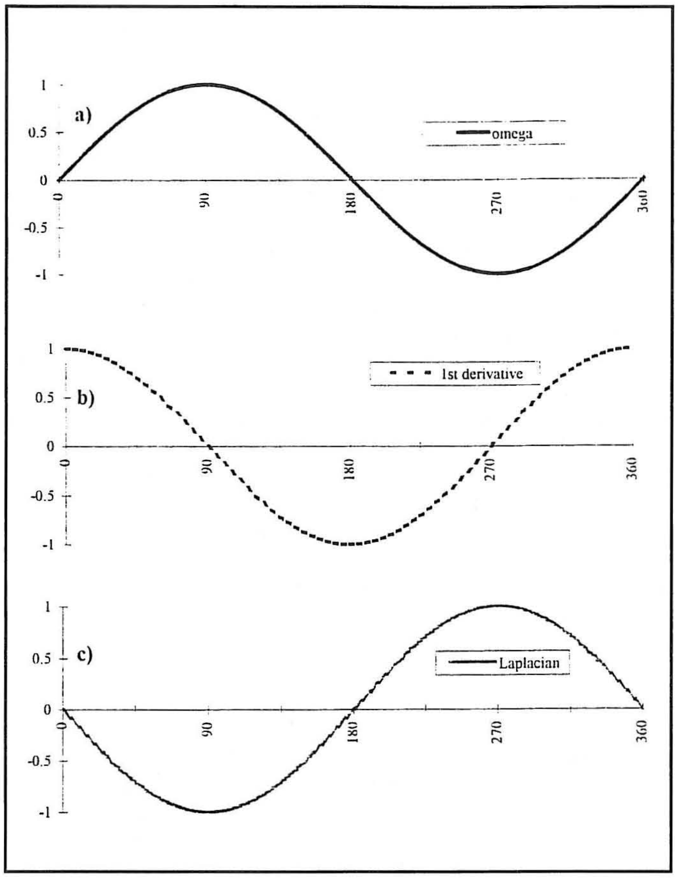

Fig. 3. Simple curves showing the relationship between omega and the Laplacian of omega in one-dimension: a) shows omega as a simple sine wave from 0° to 360°; the first derivative of omega $\partial\omega/\partial x$ b) is, therefore, a cosine wave; and c) the Laplacian of omega $\nabla^2\omega$ is represented by $-$sine x. Main point is $\nabla^2\omega$ is proportional to $-\omega$.

Hence, omega must be differentiated twice in arriving at the Laplacian. The first derivative (Fig. 3b), $\partial\omega/\partial x$, describes the instantaneous rate of change of omega with increasing x, or more simply, the slope of the original omega curve (Fig. 3a). The second derivative (Fig. 3c), $\partial/\partial x\ (\partial\omega/\partial x)$, is the Laplacian by equation (4). (Mathematically, the Laplacian describes the curvature of omega.)

The point of interest, then, is the relationship of the Laplacian of omega (Fig. 3c) and the original omega curve (Fig. 3a). In this purely sinusoidal case, the omega curve is the exact negative of the Laplacian of omega. In other words, where the Laplacian of omega is negative, omega is positive and vice-versa. Not only do the signs of the terms satisfy the proportionality in (2), but the magnitudes of the Laplacian of omega and omega are the same at each point on the respective curves (though oppositely signed). In operational terms, the right side of Fig. 3c where the Laplacian is positively valued could be representative of forcing due to positive differential vorticity advection or warm air advection (or the combination of the two). The omega response is given by the right hand side of Fig. 3a which shows omega to be negative, denoting upward motion. Hence, something familiar: positive differential vorticity advection or warm air advection forcing upward motion. Correspondingly, the left hand side could represent a relationship between a negatively valued forcing function and downward motion.

In reality, the wavelike structure of the atmosphere rarely resembles something as simple as a sine wave. Given more complicated and realistic wave patterns, the relationship between the Laplacian of omega and omega is not as straightforward. Figure 4 exhibits a slightly more complex wavelike structure of omega (solid black line) consisting of a combination of sine and cosine functions. The main difference between omega in the pure sine example and Fig. 4 is the addition of an elongated weaker area of omega centered near 180°. This simple change will help to point out some important relationships between omega and its Laplacian.

The three curves in Fig. 4 represent the same relationships to each other as the set of curves in Fig. 3. The slope of omega $(\partial\omega/\partial x)$ is shown by the dashed line and the Laplacian is depicted by the lighter solid line. In general, omega is positive from 0° to 180° and negative from 180° to 360°. The Laplacian, however, changes signs four different times, representing four distinct areas of forcing. From 0° to approximately 105°, the Laplacian is negative as expected, coincident with positive values of omega. From 255° to 360°, the Laplacian is positive in an area where omega is negative. In these two areas, (2) continues to be satisfied, i.e., the Laplacian of omega is proportional to minus omega. On the other hand, from 105° to around 255°, the Laplacian of omega has the same sign as omega, countering the critical assumption in (2).

The situation is not as unfortunate as it first appears. Consider the form of the Laplacian curve versus the omega curve. The Laplacian curve appears "noisier" than omega, or alternatively, omega looks smoother. This difference is a characteristic of the Laplacian of any field and the original field. Hence, a plane view of the Laplacian of omega at 500 mb will look noisier than the omega field itself. Moreover, the more dominant features in the Laplacian field will play the largest role in determining omega. This influence is felt not only in close proximity to the stronger features, but also in adjacent locations and levels of the atmosphere. In other words, the forcing spreads itself out spatially over a domain larger than its own.

From a physical standpoint, the relationship between forcing and the resulting omega field may be better understood by comparison with a heat source/sink problem. Consider a heat source positioned in the middle of a room. The result of this heat source is to raise the temperature at all points in the room. Small heat sinks are present which diminish the effect of the

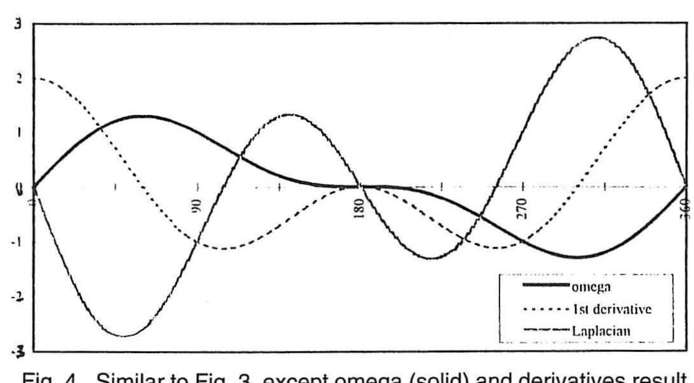

Fig. 4. Similar to Fig. 3, except omega (solid) and derivatives result from a slightly more complex function of sines and cosines. First derivative is dashed and Laplacian shown by lighter solid line. Figure shows that more dominant forcing areas determine sign of omega.

heat source, but the source is strong enough to overpower any cooling effects. The overall result of this heat source is to raise the temperature at all points within the room (not evenly). Note that the source not only affects the local temperature, but it affects the temperature at adjoining locations and levels throughout the room, even in areas where weaker heat sinks are attempting to reverse the temperature trend.

In terms of the omega equation, the heat source is represented by the Laplacian of omega (the total forcing function represented by the right hand side terms of equation (1)) and the temperature rise corresponds to the resulting omega field. As an example, consider a strong center of forcing in the middle troposphere due to positive vorticity advection (PVA, the heat source). (Assume for purposes here that PVA increases with height.) This forcing will result in upward motion not only near the PVA center, but also some distance away in the horizontal and vertical. A weak negative vorticity advection (NVA) area may be located nearby (like the heat sink above) and could help to diminish the effect of positive forcing due to the stronger PVA center. If the PVA forcing is strong enough, it will overpower the effects of the NVA and result in net upward motion, even in the vicinity of the NVA center.

What this all means for Fig. 4 is that the stronger forcing features (from approximately 0° to 105° and 255° to 360°) overpower the two weaker features in the middle of the figure, producing the resultant smoothed omega pattern. Even where the forcing should result in an opposite sign of omega (from 105° to 255°), the overpowering forcing features on the left and right hand sides result in the "correct" sign of omega.

What does this tell the forecaster? Concentrate on the large-scale, well-defined QG forcing areas when trying to use the omega equation to qualitatively evaluate the vertical motion field. The more dominant the QG forcing areas are in magnitude and scale, the more likely these features can be used to correctly determine the sign and strength of the resulting omega field.

Hopefully, the examples above provide some insight into the meaning of the Laplacian of omega and why this Laplacian is typically proportional to the negative of omega for synoptic-scale systems of interest. The bottom line is that THE LEFT HAND SIDE OF THE OMEGA EQUATION CAN BE REDUCED TO MINUS OMEGA FOR LARGE-SCALE SYSTEMS. Without this critical assumption, the simple qualitative relationships to be examined in the following sections would not exist. For a more precise and technical discussion of the potential errors associated with this assumption, refer to Appendix A in Durran and Snellman (1987).

b. Differential vorticity advection

Term B is typically the most familiar of the terms in the omega equation (eqn. 1). The portion in brackets is the vorticity advection, or more precisely, the advection of absolute geostrophic vorticity by the geostrophic wind. This more precise definition simply states that the vorticity advection can be defined by the structure of the height field (since geostrophic wind is used to define both the wind and the vorticity field). For purposes of simplicity, this term (in brackets) will, hereafter, be referred to as vorticity advection.

Since pressure decreases aloft:

$$-\frac{\partial}{\partial p} \text{ is qualitatively the same as } \frac{\partial}{\partial z}$$

where z is height. Ignoring the constant (f_0/σ) for the time being, a more qualitative representation of term B can now be written:

$$-\omega \propto -\frac{\partial}{\partial p}[(vort\ advection)] \propto \frac{\partial}{\partial z}[(vort\ advection)] \quad (5)$$

This equation states that upward motion (negative values of omega) is associated with vorticity advection increasing with height (in the Northern Hemisphere). This occurs with either positive vorticity advection (PVA) increasing with height or negative vorticity advection (NVA) decreasing with height. Conversely, downward motion (positive values of omega) is associated with vorticity advection decreasing with height, or alternatively, NVA increasing or PVA decreasing with height.

A plot of differential vorticity advection (Fig. 5) associated with a typical 500-mb disturbance crossing Ontario, Manitoba, and Minnesota displays a pattern of vorticity advection increasing with height (solid) downstream of the trough axis and vorticity advection decreasing with height (dashed) upstream of the axis. A vertical cross-section through the disturbance (not shown) confirms that these areas are associated with PVA increasing with height and NVA increasing with height, respectively. This pattern of forcing corresponds well with the expected conceptual model of rising motion downstream and subsidence upstream of a mid-tropospheric disturbance. This configuration of PVA and NVA increasing with height is common in the mid-latitudes since wind speed and vorticity, and correspondingly, the magnitude of vorticity advection tend to increase with height near the path of disturbances in the westerlies. NVA decreasing with height, however, is just as proper for forcing upward motion as PVA increasing with height. Alternatively, PVA decreasing with height is just as suitable for forcing downward motion as NVA increasing with height.

To complete the evaluation of term B, the expression (f_0/σ) that was ignored earlier must be taken into consideration. This term contains the static stability parameter (σ) in the denominator, so an inverse relationship exists between the static stability of the airmass and the degree of forcing. Hence, a given amount of differential vorticity advection will be more efficient at producing vertical motions in an airmass that is less statically stable than in an environment of highly statically stable air. With this in mind, a measure of static stability should be used in conjunction with the pattern of differential vorticity advection (or even for simple 500-mb vorticity advection) to properly evaluate the forcing.

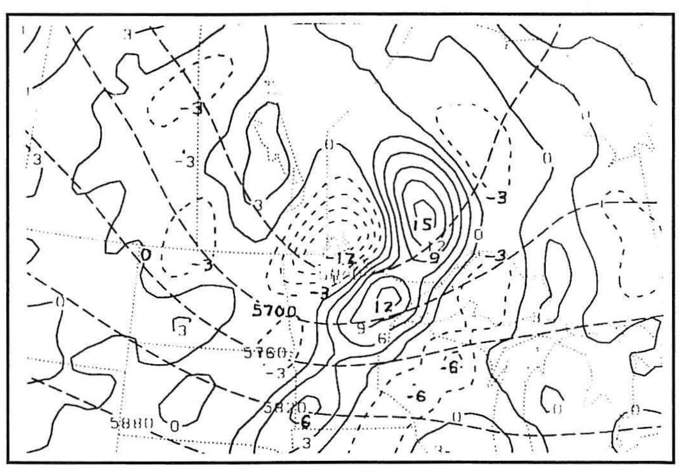

Fig. 5. Differential vorticity advection from Eta model valid 1200 UTC 22 July 1996 near a 500-mb short-wave trough calculated in a layer from 450–550 mb. Contour interval is 3×10^{-17} Pa^{-1} s^{-3}. Solid (dashed) contours are positive (negative) values and imply forcing for upward (downward) motion. Height (long-dashed) field contoured every 60 meters.

To illustrate this point, the static stability ($\partial\theta/\partial p$) corresponding to the previous plot of differential vorticity advection (Fig. 5) is exhibited in Fig. 6. (This term will be a reasonable approximation to the static stability parameter, σ). Note that the static stability is lowest in southeastern Manitoba (dashed in Fig. 6), near the center of forcing where NVA is increasing with height (dashed in Fig. 5). Static stability generally increases to the southeast into the Great Lakes region, toward the area where PVA is increasing with height (Figs. 5 and 6). This pattern indicates that the forcing for downward motion centered on the Manitoba/Ontario border is likely greater than the corresponding forcing for upward motion centered further east over southwest Ontario. Though both centers have approximately the same magnitude of differential vorticity advection, the lower static stability where NVA is increasing with height results in better forcing.

Obviously, term B is related to the historical usage of 500-mb vorticity advection as a proxy for vertical motion. Take away the static stability term (f_0/σ) and the vertical derivative ($\partial/\partial p$) and term B simplifies to vorticity advection at some level (such as 500 mb). Why 500 mb? One reason is that forcing provided by differential vorticity advection is better correlated with omega in the mid-levels of the troposphere. Additionally, since 500 mb is close to the level of non-divergence, vorticity is nearly conserved at this level. In other words, the vorticity field moves along with the flow and can be followed more easily with time.

Qualitatively, the sign of the forcing can usually be determined correctly by using either 500-mb vorticity advection, differential vorticity advection, or the full term B (ignoring the effects of temperature advection for now). Note the term "usually." Quantitatively, the relative strength of the forcing may be quite different amongst the different methods and may differ significantly depending on the chosen level or layer. For example, vorticity advection at 500 mb (Fig. 7) for the case shown in Fig. 5 suggests three different centers of PVA (two in Minnesota and one in southwest Ontario). The layer from 450–550 mb chosen for display in Fig. 5 exhibits two different centers, but the overall pattern is about the same. By using a deeper layer (300–700 mb, Fig. 8) to calculate differential vorticity advection, it becomes apparent that the PVA center (+12 units center in Fig. 8) in southwest Ontario is a deeper and more well-defined feature. Spatial cross-sections (not shown) verify that the forcing (due only to term B) is approximately twice as strong in southwest Ontario as in Minnesota. Thus, it behooves the forecaster to investigate the forcing over more than one level or layer if possible.

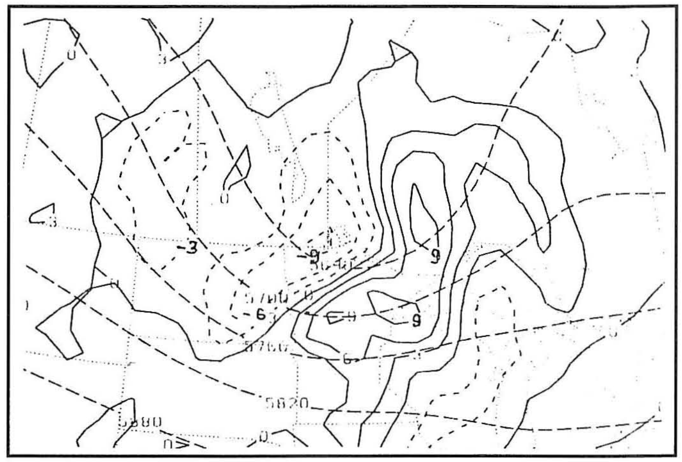

Fig. 7. 500-mb absolute vorticity advection matching the time period in Fig. 5. Contour interval is 3×10^{-9} s^{-2}. Solid (dashed) contours are positive (negative) values and imply forcing for upward (downward) motion. Height (long-dashed) field contoured every 60 meters.

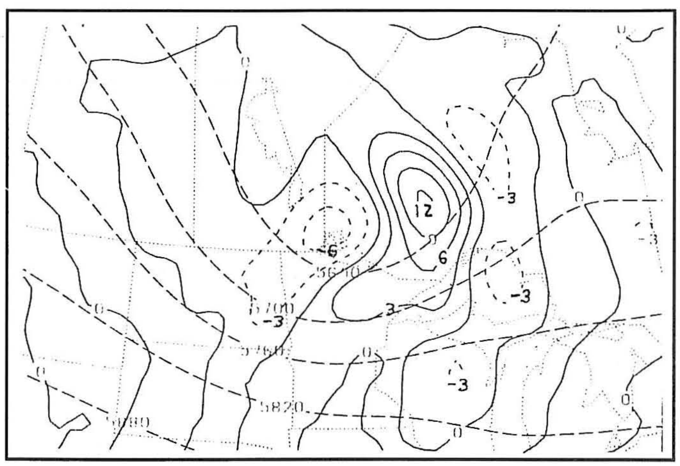

Fig. 8. Differential vorticity advection calculated from 300–700 mb. Contours and time period same as in Fig. 5.

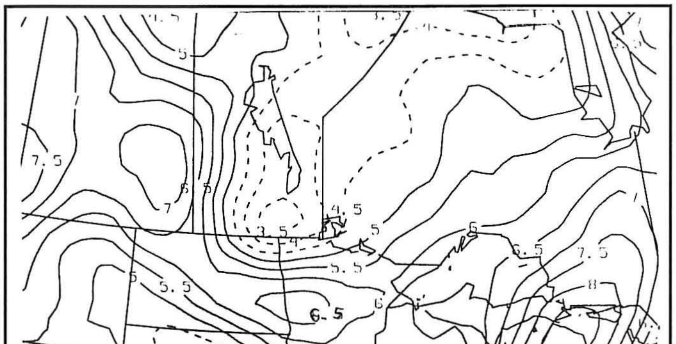

Fig. 6. Static stability ($\partial\theta/\partial p$) calculated in 450–550 mb layer matching the time period in Fig. 5. Contour interval is 0.5×10^{-4} K Pa^{-1}. Lower static stability represented by dashed contours.

c. Laplacian of temperature advection

With a past fixation on vorticity advection, forecasters have often overlooked term C. In its form in equation (1), term C is actually the Laplacian of thickness advection. Since thickness can be shown to be proportional to the average temperature of a layer, term C can be rewritten as:

$$-\omega \propto -\frac{1}{\sigma}\nabla^2[-\vec{V}_g \cdot \nabla(\bar{T})] \qquad (6)$$

and can now be thought of as the Laplacian of the mean temperature advection of a layer. The Laplacian can be treated in a similar fashion to the Laplacian of omega in section *a,* assuming the temperature advection field is considered wavelike. This assumption reduces (6) to the following qualitative statement:

$$-\omega \propto -\frac{1}{\sigma}\nabla^2[\textit{temp advection}] \propto +\frac{1}{\sigma}[\textit{temp advection}] \qquad (7)$$

which says that warm advection is associated with upward motion and cold advection with downward motion.

It is interesting to note that if term B is ignored in equation (1), the resulting equation describes the relationship of the three-dimensional Laplacian of a variable on the left-hand-side with the two-dimensional Laplacian of a variable on the right-hand-side:

$$\underbrace{\left(\nabla_h^2 + \frac{f_0^2}{\sigma}\frac{\partial^2}{\partial p^2}\right)}_{\textit{3-d Laplacian}}\omega = -\frac{1}{\sigma}\underbrace{\nabla^2}_{\textit{2-d Laplacian}}\left[-\vec{V}_g\cdot\nabla\left(-\frac{\partial\phi}{\partial p}\right)\right] \qquad (8)$$

Hence, it can be argued that omega resulting from term C is more closely related to the temperature advection field than the Laplacian of the temperature advection.

An example of the temperature advection field, corresponding to the vorticity advection pattern in Fig. 5, is presented in Fig. 9. (The Laplacian of this field is shown in the next section.) Cold advection (dashed), and hence downward motion, is suggested across southwest Ontario, stretching back into portions of northern Minnesota to southeast South Dakota. A band of warm advection (solid), forcing upward motion, is seen to the northwest from the southwestern tip of Ontario into southeast North Dakota. Another more disorganized band of warm advection exists just ahead of the cold advection area mentioned above located from central Ontario into the Great Lakes region. A quick comparison of Figs. 5 and 9 easily shows areas of conflict between the two forcing terms (to be discussed in the next section).

The forcing provided by the temperature advection term (term C) is also inversely proportional to the static stability parameter (note σ in the denominator in eqs. 6 and 7). Given a constant value of temperature advection (or the Laplacian), where static stability is lower, the magnitude of forcing will be higher. Where static stability is higher, the degree of forcing will be diminished. Hence, the temperature advection term should also include some measure of static stability for quantitative evaluation of the forcing.

As with differential vorticity advection, the sign of the forcing can usually be determined correctly by using either the temperature advection, the Laplacian of temperature advection, or the full term C. If the objective is to infer omega resulting from term C in isolation (ignoring the vorticity advection term), it is best to simply use temperature advection. If the objective is to compute the full forcing function on the right-hand-side of (1), the full term C is necessary. Once again, results may vary depending on the level or layer chosen.

d. Total forcing

To produce the total forcing for vertical motion provided by a QG disturbance, both the differential vorticity advection and temperature advection terms must be combined. Unfortunately, in many cases, the forcing provided by these terms tends to be of the opposite sign. To illustrate this point, compare the Laplacian of temperature advection (Fig. 10) with the differential vorticity advection (Fig. 5) for the case presented in the previous sections. The implied areas of vertical motion almost seem 180° out of phase. For instance, the maximum in differential vorticity advection over southwest Ontario (solid, 15 unit center in Fig. 5) is nearly collocated with an area of suggested cold air advection (dashed, −15 unit center in Fig. 10). This implies upward motion forced by the differential vorticity advection term and downward motion forced by the temperature advection term. Since both figures (5 and 10) have been produced with the same units, the terms can simply be added together to arrive at the correct forcing (Fig. 11). For those who continue to rely mostly on vorticity advection, note that the total forcing certainly has some significant differences compared to the vorticity advection pattern (Figs. 7 and 11), although the general configuration is qualitatively similar. This similarity suggests that term B is dominant in the mid-levels for this system at this point in time. Note that the contributions from terms B and C to the total forcing can vary considerably, depending on the selected level, layer, or evolutionary phase of the system.

Historically, terms B and C in equation (1) have been either qualitatively evaluated from centrally generated charts or calcu-

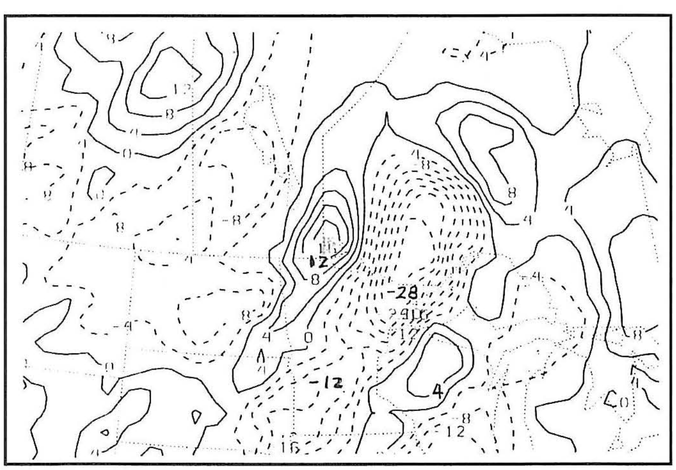

Fig. 9. Temperature advection at 500 mb. Contour interval is 2×10^{-5} K s^{-1}. Solid (dashed) contours represent warm (cold) advection implying upward (downward) vertical motion.

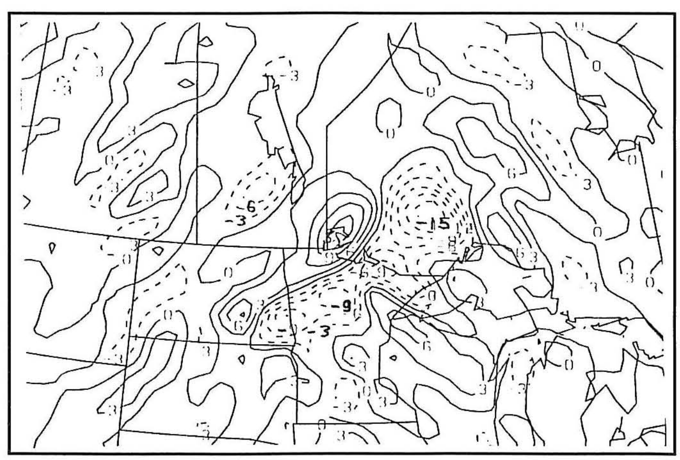

Fig. 10. Laplacian of thickness advection calculated in 450–550 mb layer corresponding to period in Fig. 5. Contour interval is 3×10^{-17} Pa^{-1} s^{-3}. Solid (dashed) contours are positive (negative) values and imply forcing for upward (downward) motion.

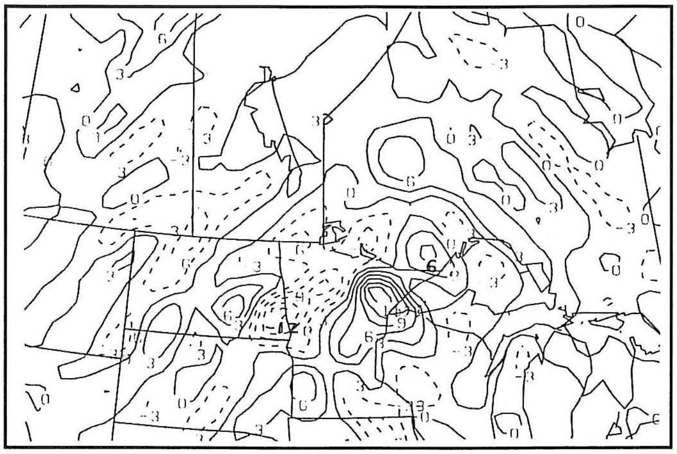

Fig. 11. Total forcing for omega produced by addition of differential vorticity advection and Laplacian of thickness advection shown in Figs. 5 and 10, respectively. Contours and time period same as in Fig. 5.

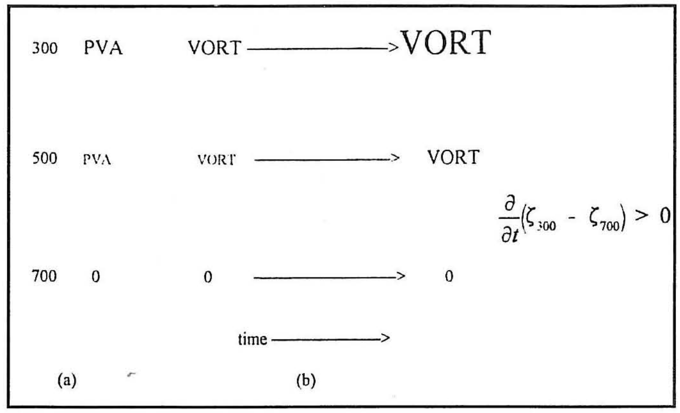

Fig. 12. Conceptual diagram displaying results of differential vorticity advection (geostrophic forcing). (a) shows PVA increasing with height from 0 at 700 mb to large values at 300 mb. (b) exhibits the temporal trend of vorticity resulting from vorticity advection increasing with height. Vorticity increases at a faster rate aloft, so difference in vorticity between low and high levels increases with time.

lated using simplifications of the original terms. Either way, evaluation of the total forcing was difficult due to the opposing terms. This problem was partially the impetus for the development of alternative derivations of the omega equation to be discussed in future articles (Q-vectors for instance). Resources are now adequate in most operational settings to evaluate the total forcing of the right hand side and determine which physical mechanism is dominant—vorticity or temperature advection.

3. A Balancing Act

Geostrophic balance is a fairly easy concept to grasp, i.e., the geostrophic wind is determined by the height field. Hydrostatic balance is also a rather simple idea—the thickness of a layer is proportional to the mean temperature of that layer (as in the use of 1000–500 mb thickness fields as an approximation for mean layer temperature). These two balance states can also be combined into the thermal wind relationship which says that the horizontal temperature gradient is related to the vertical shear of the geostrophic wind. On the synoptic-scale, the atmosphere is always nearly in a balanced state which satisfies these relationships. It is the disturbances or the imbalances, however, which tend to ''force'' the interesting weather.

QG theory provides a very nice conceptual framework in order to understand the atmosphere in terms of disturbances and the tendency to return to a balanced state. Consider an area of PVA increasing with height and negligible temperature advection (Fig. 12a). Assuming no other effects, how does this forcing affect the vorticity distribution with time? To answer this question, recall the QG vorticity equation:

$$\frac{\partial \zeta_g}{\partial t} = -\vec{V}_g \cdot \nabla(\zeta_g + f) + f_0\frac{\partial \omega}{\partial p} \tag{9}$$

change in vorticity with time at a point	*advection of vorticity*	*divergence*

which says that the vorticity (at a point) can change with time by only two processes: advection of vorticity and divergence. Obviously, discounting divergence effects (for now), vorticity will increase with time at each level. The vorticity, however, increases aloft at a faster rate than below (Fig. 12b), leading to the relationship:

$$\frac{\partial}{\partial t}(\zeta_{300} - \zeta_{700}) > 0 \tag{10}$$

which says that the vorticity difference between 300–700 mb is increasing with time (at a point).

With the vorticity field increasing with time, it is obvious that the height field would have to change in some way to keep the atmosphere geostrophic. Since vorticity (geostrophic) is related to the height field by:

$$\zeta_g \propto \nabla^2\Phi \tag{11}$$

where Φ is the geopotential height, (10) can be rewritten:

$$\frac{\partial}{\partial t}\nabla^2(\Phi_{300} - \Phi_{700}) > 0 \tag{12}$$

Assuming the thickness field $(\Phi_{300} - \Phi_{700})$ is wavelike, (12) becomes:

$$\frac{\partial}{\partial t}(\Phi_{300} - \Phi_{700}) < 0 \tag{13}$$

Thus, the necessary adjustment for increasing the vorticity difference with time (10) is to decrease the thickness of the layer. This adjustment seems logical since vorticity is increasing faster aloft than below, so heights should be falling faster aloft than below.

To keep the atmosphere in hydrostatic balance, the mean temperature of the layer (300–700 mb) must decrease along with the thickness. If the temperature remains the same and the thickness does not decrease, then the atmosphere cannot move back toward geostrophic balance (the winds and vorticity would be out of balance with the height field). What must happen to keep the atmosphere BOTH geostrophic and hydrostatic? The adjustment in QG theory is to provide a secondary circulation. The vertical branches of this circulation are simply the omega fields diagnosed from the omega equation. The adjustment in this case is shown in Fig. 13a.

Upward motion that is associated with differential vorticity advection per the omega equation helps balance the atmosphere in two ways. First, the upward motion results in adiabatic cooling of the layer which moves the atmosphere back toward hydrostatic balance (remember the mean temperature needed to be lower to ''agree'' with the changing thickness field). Secondly, the omega distribution in Fig. 13a results in a vertical distribution of divergence which helps to dampen the geostrophic disturbance. Divergence occurs aloft which from (9) tends to decrease the vorticity with time. Convergence develops below which tends to increase the vorticity with time. This configuration results in:

$$\frac{\partial}{\partial t}(\zeta_{300} - \zeta_{700}) < 0 \qquad (14)$$

which counters the trend from the geostrophic disturbance shown in (10). The response of omega is shown conceptually in Fig. 13b.

Looking at (10) and (14), it is obvious that two processes (differential vorticity advection effects and the divergence effects from the omega field) are occurring which oppose one another. It turns out that the effects due to differential vorticity advection dominate so that for the overall system:

$$\frac{\partial}{\partial t}(\zeta_{300} - \zeta_{700}) > 0 \qquad (15)$$

which necessitates a thickness change:

$$\frac{\partial}{\partial t}(\Phi_{300} - \Phi_{700}) < 0 \qquad (16)$$

which leads to a need for a cooling of the airmass through omega. An important point, here, is that the omega pattern must play two roles which are not independent. A given omega pattern is related to the thickness change through adiabatic cooling, but also partially determines this same thickness change through its divergence pattern. Obviously, a balance must be achieved.

It is important to note that all of the above processes occur simultaneously. The omega field is an instantaneous response to the geostrophic disturbance. The geostrophic disturbance tries to move the atmosphere out of balance (geostrophic or hydrostatic) while the non-geostrophic response (omega) tries to restore the atmosphere to a balanced state. The magnitude of omega will be just enough to produce this balance. Relatedly, the relationships discussed above say nothing about how the various fields will evolve. A given magnitude of differential vorticity advection cannot be used to predict the future state of the omega field. It can only be used to diagnose the current state of the omega field.

So far, the discussion has centered around term B, differential vorticity advection. The same balance ideas pertain to the temperature advection term (C) as well. The details are described elsewhere (see Bluestein 1992 or Holton 1992), but it is easy to see how the effect of a temperature advection disturbance could be reduced by adiabatic vertical motion. For example, consider the case of forcing provided by warm advection. The temperature advection, or geostrophic disturbance, tends to warm the layer in question. Upward vertical motion is associated with this warm advection area which counters the warming by adiabatic cooling due to this lift. Once again, the omega field produced as a response to the temperature advection forcing should be of sufficient magnitude to keep the atmosphere hydrostatic and geostrophic.

4. Level of QG Forcing

A common question asked by many forecasters is ''What level or layer is best to evaluate the QG forcing?'' The answer is not straightforward. What is clear is that differential vorticity advection typically dominates the forcing in the middle-upper troposphere (Figs. 5, 10, and 11 to see an example). The temperature advection term tends to become more important in the lower levels (Figs. 14, 15, and 16). The dominance of each term also depends on the evolution of a system. For example, in traditional cyclogenesis theory, one would expect differential vorticity advection to dominate in the early stages before cyclogenesis occurs. As the cyclone spins up, temperature advection

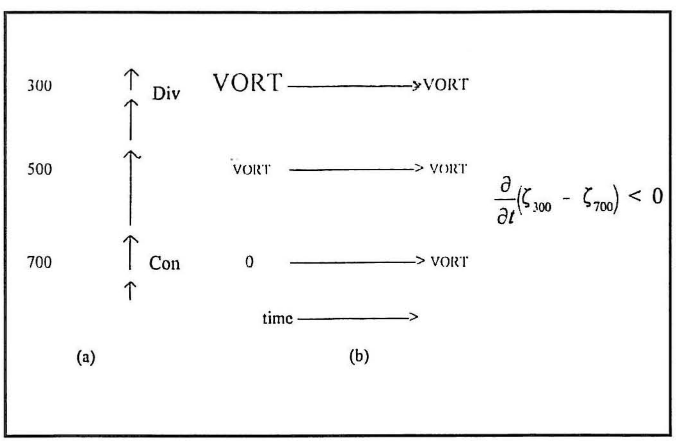

Fig. 13. Conceptual diagram showing the instantaneous non-geostrophic response to processes shown in Fig. 12. (a) is resulting vertical distribution of omega with divergence (convergence) of the wind aloft (below). (b) exhibits the temporal trend of vorticity due to the pattern shown in (a). Divergence aloft tends to decrease vorticity with time and convergence below tends to increase vorticity with time, so difference in vorticity between low and high levels decreases with time.

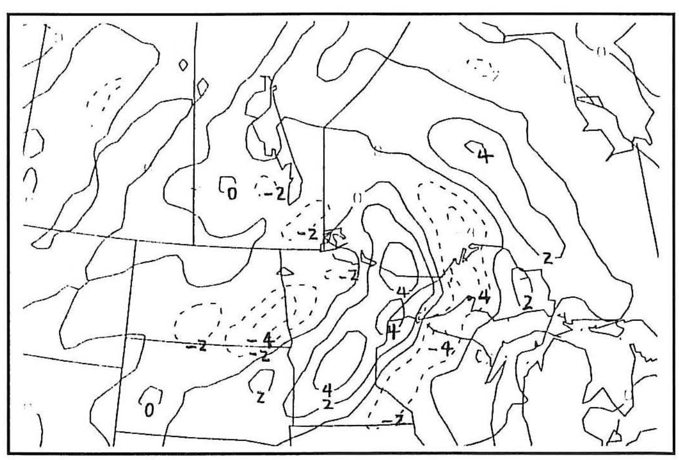

Fig. 14. Differential vorticity advection calculated in layer 700–800 mb. Contours and time period same as in Fig. 5, except interval is 2×10^{-17} Pa^{-1} s^{-3}.

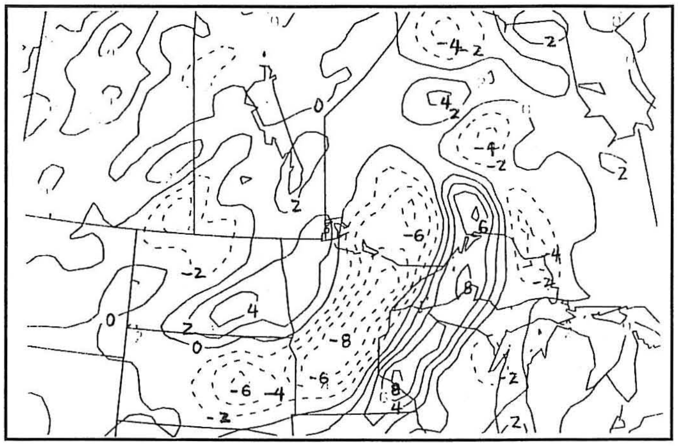

Fig. 15. Laplacian of thickness advection calculated in layer 700–800 mb. Contours and time period same as in Fig. 5, except interval is 2×10^{-17} Pa^{-1} s^{-3}.

Fig. 16. Total forcing for omega in layer 700–800 mb computed by adding terms in Figs. 14 and 15. Contours and time period same as in Fig. 5, except interval is 2×10^{-17} Pa^{-1} s^{-3}. Comparison with Figs. 14 and 15 reveal thickness advection term more dominant.

terms would become increasingly important. These relationships are likely no surprise to the operational forecaster.

What is not as well understood is the relationship between the forcing at a single level or shallow layer and the vertical distribution of omega. A common misconception is that the forcing at one level contributes to the omega field only at that level. As shown in the discussion of the Laplacian (section 2a), omega at one level may be better related to the forcing at another level. Thus, the 700-mb omega pattern may be better correlated to the 500 mb forcing than the 700 mb forcing. It turns out that the relationship between omega and the right hand side forcing at the same level is best defined in the middle troposphere. For this reason, the total forcing should generally be evaluated near 500 mb or in the mid-troposphere from around 700–300 or 600–400 mb. This is not to say that low-level temperature advection fields are unimportant. If the low-level temperature advection field is dominant, and mid-upper level vorticity advection is weak, 700-mb temperature advection may be closely correlated to 700-mb omega. In a case like this, a layer centered around 700 mb may be best. A more detailed and mathematical discussion of these issues can be found in Appendix A in Durran and Snellman (1987) or the Appendix in Trenberth (1978).

5. Summary

Terms such as vorticity and temperature advection are very common in discussions of forecast reasoning and almost always implicitly refer to the forcing of the vertical motion field. Forecasters obviously are quite familiar with these forcing terms and, whether they realize it or not, the right hand side of the omega equation. Hence, portions of this article are simply review. On the other hand, it is the author's opinion that most operational forecasters know much less about the relationship of omega to the forcing functions or the concept of the balance state of the atmosphere under which QG theory operates. In other words, HOW is omega related to the forcing functions? WHY is omega related to the forcing terms? How does differential vorticity advection translate into upward vertical motion from a physical standpoint? The discussion on the Laplacian of omega and the balance state of the atmosphere attempts to answer these questions.

With this understanding in hand, the reader will have the background to comprehend alternative approaches to the traditional equation, namely the Q-vector (Hoskins et al. 1978) and Trenberth (1978) formulations. In these approaches, the forcing functions on the right hand side of the omega equation are combined into one forcing term, eliminating the need for evaluation of two potentially opposing terms. These alternative approaches will be the subject of the next article.

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